

AD-A173 513

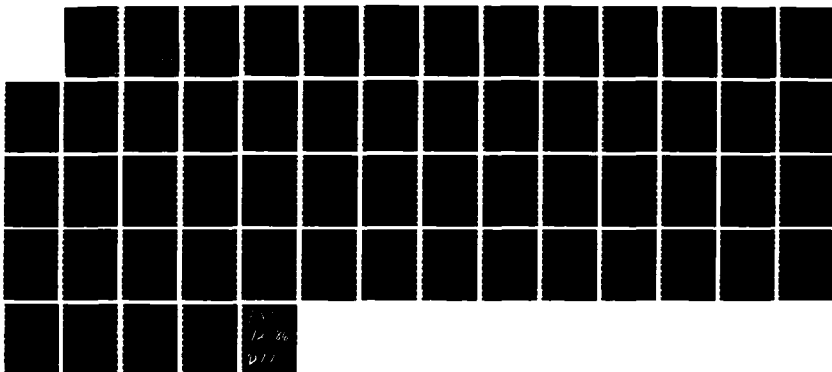
DIMENSIONAL REDUCTION FOR NONLINEAR BOUNDARY VALUE
PROBLEMS(U) MARYLAND UNIV COLLEGE PARK INST FOR
PHYSICAL SCIENCE AND TECHNOLOGY S JENSEN ET AL AUG 86
BN-1055 N00014-85-K-0169

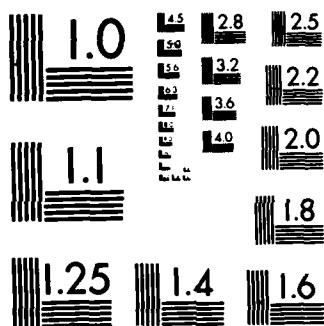
1/1

UNCLASSIFIED

F/G 12/1

NL





XEROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

15

Technical Note BN-1055

AD-A173 513

DIMENSIONAL REDUCTION FOR NONLINEAR BOUNDARY VALUE PROBLEMS

by

Soren Jensen
University of Aalborg
Denmark

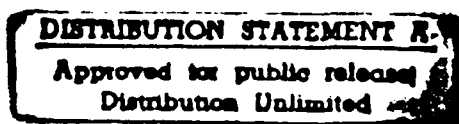
and

University of Maryland, Baltimore County, Maryland 21201
U.S.A.

and

I. Babuška
Institute for Physical Science and Technology
University of Maryland, College Park, Maryland 20742
U.S.A.

DTIC FILE COPY



DTIC
ELECTE
OCT 23 1986
S B

August 1986



UNIVERSITY OF MARYLAND

86 10 17 007

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|--|---|--|
| 1. REPORT NUMBER Technical Note BN-1055 | 2. GOVT ACCESSION NO. ADA173513 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) Dimensional Reduction for Nonlinear Boundary Value Problems | | 5. TYPE OF REPORT & PERIOD COVERED Final life of the Contract |
| 7. AUTHOR(s) Soren Jensen and I. Babuska (University of Aalborg, Denmark) | | 6. PERFORMING ORG. REPORT NUMBER |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742 | | 8. CONTRACT OR GRANT NUMBER(s) ONR N00014-85-K-0169 ONR N00014-85-K-0169 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS Department of Naval Research Office of Naval Research Arlington, VA 22217 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 12. REPORT DATE August 1986 |
| | | 13. NUMBER OF PAGES 55 |
| | | 15. SECURITY CLASS. (of this report) |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |
| 16. DISTRIBUTION STATEMENT (of this Report) Approved for public release: distribution unlimited | | |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) | | |
| 18. SUPPLEMENTARY NOTES | | |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Nonlinear elliptic equations, dimensional reduction, asymptotic analysis, numerical treatment of partial differential equations. | | |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The paper elaborates on the method of dimensional reduction for nonlinear problems which transforms the 2 dimensional problem into a system of ordinary differential equations in an optimal way. Numerical example illustrates the approach. | | |

- 1 -

Dimensional Reduction
for Nonlinear Boundary Value Problems

by

Soren Jensen¹⁾

and

Ivo Babuška²⁾

List of contents

1. Introduction.
2. Notation and model problem.
3. The Galerkin approach to dimensional reduction.
4. Asymptotic expansions (for the selection of basis functions).
5. The selection of basis functions.
6. The optimality of above selection of basis functions (at fixed N).
7. Optimal convergence for increasing number of reduced models
(at fixed ϵ).
8. Computational results.

1) Supported by the Office of Naval Research under Contract N 00014-85-K-0169,

University of Aalborg, Denmark, and University of Maryland Baltimore County.

2) Partially supported by the Office of Naval Research under Contract N 00014-85-K-0169.

1. Introduction. In this paper we describe a systematic approach to the dimensional reduction of nonlinear, monotone boundary value problems. Our work is motivated by problems arising from considering "thin" mechanical structures such as beams, arches, plates and shells, cf. [2] and [26]. Let $\Omega^\epsilon =]0; 1[x] - \epsilon; \epsilon[$ be a domain in \mathbb{R}^2 . We present an efficient procedure to produce approximate solutions to the above boundary value problems defined over Ω^ϵ . Led by intuition, we expect that the solution can be approximated well by linear combinations

$$\sum_{j=0}^N c_j(x_1) \psi_j(x_2/\epsilon) ; \quad x_1 \in [0;1], x_2 \in [-\epsilon, \epsilon].$$

We select the basis- or "Ansatz" functions $\{\psi_j\}_{j=0}^\infty$ so as to obtain maximal order of convergence as ϵ tends to zero. $\{\psi_j\}_{j=0}^N$ are found as solutions to Galerkin systems of ordinary differential equations. We consider three asymptotic ranges of loads (infinite, finite, and zero limit) and the selection of "Ansatz" functions is shown to be independent thereof. This is of course of major importance for practical computations. Estimates for the order of convergence as N tends to infinity are established (as they must be, since ϵ is never infinitesimally small in practical applications and since a required accuracy is often not obtainable by taking more terms in the asymptotic expansion). The span of $\{\psi_j\}_{j=0}^N$ is shown to be optimal in the sense of N -widths in approximation theory.

This method differs from another method of dimensional reduction widely used in structural mechanics, that of asymptotic expansion in ϵ , see Friedrichs [8], Friedrichs and Dressler [9], Reiss and Locke [28], Gol'denweizer [11], and Ciarlet and Destuynder [6]. The latter method is



limited in its practical scope since it deals with rough input data by introducing boundary - and/or interior layer expansions which often are difficult to obtain without extensive analysis dependent upon the particular case at hand. This limits the generality of the latter approach. We propose to deal with such nonsmoothness by increasing N near the layer.

The aforementioned characteristics of our approach are practically feasible as witnessed through our computational experience in the last section of the paper.

We remark that this work was inspired by the papers of Vogelius and Babuska [30, 31, 32] on dimensional reduction for linear, elliptic boundary value problems.

2. Notation and model problem.

Let A be an operator of the form

$$(1) \quad Au(v) = \int_{\Omega^\epsilon} F(|Du|^2) Du \cdot Dv \, dx,$$

where $x = (x_1, x_2) \in \Omega^\epsilon =]0, 1[x] - \epsilon, \epsilon[$. Denote by $\Gamma_\pm^\epsilon = [0, 1] \times \{\pm\epsilon\}$ and $[0, 1] \times \{-\epsilon\}$ respectively, and by $\Gamma_0^\epsilon = (\{0\} \times]-\epsilon, \epsilon[) \cup (\{1\} \times]-\epsilon, \epsilon[)$, so that $\partial\Omega^\epsilon = \Gamma_+^\epsilon \cup \Gamma_-^\epsilon \cup \Gamma_0^\epsilon$ disjointly. Let $Du = (\partial u / \partial x_1, \partial u / \partial x_2)$ and $|Du| = (\sum_{i=1}^2 (\partial u / \partial x_i)^2)^{1/2}$.

Let for some $n \in \mathbb{Z}_+$,

$$(2) \quad F(t) = 1 + t^n, \quad \forall t \in \mathbb{R}_+ \setminus \{0\},$$

for the sake of explicitness. What is to follow could be extended to other choices of F without much imaginative effort, cf. [27].

We consider the minimization problem: Find u^ϵ such that

$$(3) \quad u^\epsilon \in W_{(0)}^{1, 2n+2}(\Omega^\epsilon) \text{ and } E(u^\epsilon) = \inf_{v \in W_{(0)}^{1, 2n+2}} E(v),$$

where

$$W_{(0)}^{1, 2n+2}(\Omega^\epsilon) = W^{1, 2n+2}(\Omega^\epsilon) \cap \{v : v|_{\Gamma_0^\epsilon} = 0\}, \text{ and}$$

$$E(v) = B(u) - G(v),$$

where the Fréchet derivative of B at u is Au , and where $G \in (W_{(0)}^{1, 2n+2})^*$

- the dual space of $W_{(0)}^{1, 2n+2}$, is defined for all $\mu \in \mathbb{R}$ and all

$\beta \in L^{2n+2}(0, 1)$ through

$$G(v) = \int_0^1 \beta(x_1) \epsilon^{1-\mu} [v(x_1, \epsilon) + v(x_1, -\epsilon)] \, dx_1.$$

Here, μ is introduced in order to investigate three asymptotic ranges of "loads" on Γ_{\pm}^{ϵ} . According to whether $\mu > 0$, $\mu = 0$, or $\mu < 0$, we have an infinite, finite, or zero "load" (ϵ^1 can be viewed as a scaling constant; it is the Jacobian of the transformation $(x_1, x_2) \rightarrow (\zeta, \eta) = (x_1, \frac{x_2}{\epsilon})$).

The minimization problem (3) is formally equivalent to the homogeneous Neumann/Dirichlet boundary value problem

$$\begin{aligned} (4) \quad & -D \cdot (F(|Du^{\epsilon}|^2)) Du^{\epsilon} = 0 \quad \text{in } \Omega^{\epsilon}, \\ & u^{\epsilon} = 0 \quad \text{on } \Gamma_0^{\epsilon}, \\ & F(|Du^{\epsilon}|^2) \partial u^{\epsilon} / \partial \nu = \beta \epsilon^{1-\mu} \quad \text{on } \Gamma_{\pm}^{\epsilon}, \end{aligned}$$

where ν is the outward normal, and furthermore, (3) is equivalent to finding $u^{\epsilon} \in W_{(0)}^{1,2n+2}(\Omega^{\epsilon})$ satisfying

$$(5) \quad \forall v \in W_{(0)}^{1,2n+2}(\Omega^{\epsilon}), \quad Au^{\epsilon}(v) = G(v).$$

Because E is strictly convex and $\lim_{|v| \rightarrow +\infty} E(v) = \infty$, (3) and (5) have a unique solution $u^{\epsilon} \in W_{(0)}^{1,2n+2}(\Omega^{\epsilon})$, cf. Visik [29], Morrey [25], Glowinski & Marocco [10], or Ciarlet [7]; we define $|v| = (\int \int_{\Omega^{\epsilon}} |Dv|^{2n+2} dx)^{1/2n+2}$ (using Friedrich's inequality).

Finally, we scale the problem as follows

$$(6) \quad u^{\epsilon}(x_1, x_2) = \tilde{u}^{\epsilon}(x_1, x_2/\epsilon), \quad (x_1, x_2) \in \Omega^{\epsilon}.$$

Denote by $(\xi, \eta) = (x_1, x_2/\epsilon)$, $D_{\epsilon} = (\partial/\partial \xi, \epsilon^{-1} \partial/\partial \eta)$, and

$\Omega =]0,1[x] - 1,1[$. (5) becomes: $\forall v \in W_{(0)}^{1,2n+2}(\Omega)$

$$(7.a) \quad \int_{\Omega} F(|D_{\epsilon} \tilde{u}^{\epsilon}|^2) D_{\epsilon} \tilde{u}^{\epsilon} \cdot D_{\epsilon} v \, \epsilon \, d\xi d\eta \\ = \epsilon \int_0^1 \beta(\xi) [v(\xi, 1) + v(\xi, -1)] \, \epsilon^{-\mu} d\xi.$$

In the remainder of the paper, we shall denote \tilde{u}^{ϵ} by u , and the left hand side of (7) by $Au(v)$ and the right hand side by $G(v)$ for any $u, v \in W_{(0)}^{1, 2n+2}(\Omega)$. Thus (7) can be written (find $u \in W_{(0)}^{1, 2n+2}(\Omega)$ such that)

$$(7.b) \quad \forall v \in W_{(0)}^{1, 2n+2}(\Omega), \quad Au(v) = G(v).$$

Also, $W_{(0)}^{1, 2n+2}(\Omega)$ is endowed with the norm $\|v\| = (\int_{\Omega} |D_{\epsilon} v|^{2n+2} \, \epsilon d\xi d\eta)^{1/2n+2}$ (again with the use of Friedrich's inequality; here $|\Omega| = 2$ independent of ϵ). Note that $\|\tilde{v}^{\epsilon}\|_{\Omega} = \|v\|_{\Omega^{\epsilon}}$.

The scalar problem (4) is a mathematical model of the physical models of antiplane shear in finite elasticity, see Gurtin & Temam [12], Abeyaratne [1], and Knowles [16, 17], and the torsion problem, see Kachanov [15] and Langenbach [20]. We are trying to express a nonlinear constitutive (stress-strain) law, so this covers finite elasticity only insofar as there exists some inherent scalar quantity in our problem. In [1] e.g., the stored energy function W is assumed to depend only on the first invariant of the left Cauchy-Green strain tensor: $3 + |Du|^2$, where the inherent scalar is $u(x_1, x_2)$ the x_3 -displacement of a prismal body along the x_3 -axis, see [1] sections [2] and [5]. In the absence of body forces, the problem is described by $D(W'(3 + |Du|^2))Du = 0$ in Ω with boundary conditions.

The most direct interpretation of (4) is that of the (stationary) heat equation with a nonlinear thermal conductivity (dependent upon the temperature gradient).

3. The Galerkin approach to dimensional reduction.

In this section we present some known results. We will consider the Galerkin approach to solve (7) and (2) on a proper subspace V_N which is characterized by the Ansatz functions $\{\psi_j\}_{j=0}^N$:

$$(8) \quad V_N = \left\{ \sum_{j=0}^N c_j(\zeta) \psi_j(\eta) : \{c_j\} \in W_{00}^{1,2n+2}(0,1) \right\}.$$

More precisely, define u_N to be the solution in V_N of

$$(9) \quad Au_N(v) = G(v), \quad \forall v \in V_N,$$

where A and G were defined in the last section's closing paragraphs. We note that the problem (9) is equivalent to the minimization problem (3) restricted to V_N , which had a unique solution (see references mentioned in connection with (3), same argument).

The operator $A: W_{(0)}^{1,2n+2}(\Omega) \rightarrow (W_{(0)}^{1,2n+2}(\Omega))^*$ is

i) strongly monotone, i.e., $\exists \chi: \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}$, strictly increasing with $\chi(0)=0$ and $\lim_{t \rightarrow \infty} \chi(t) = \infty$, such that $(Au - Av)(u - v) \geq \chi(\|u - v\|) \|u - v\|$, $\forall u, v \in W_{(0)}^{1,2n+2}(\Omega)$, and

ii) Lipschitz-continuous for bounded arguments, i.e., $\forall r > 0 \exists \Gamma(r)$ such that if $\|u\|, \|v\| \leq r$, then $\|Au - Av\|^* \leq \Gamma(r) \|u - v\|$; cf. Ciarlet [7], sec. 5.3.

This is proved easily along the lines there which are based upon the work of Glowinski & Marocco [10] and could be generalized to hold for polynomials F that satisfy $(sF(s^2) - tF(t^2))/(s-t) > 0$, $\forall s, t > 0, s \neq t$, see [14]. It is also known (cf. Ciarlet [7] or Glowinski & Marocco [10]) that when A satisfies

these two properties, then there exists C independent of V_N such that

$$\chi(|u - u_N|) \leq C \inf_{v_N \in V_N} |u - v_N|.$$

In our case, $\exists \alpha > 0: \chi: t \rightarrow \alpha t^{2n+1}, \forall t \geq 0$. This is a nonlinear generalization of Cea's lemma.

Since C in the last inequality depends on ϵ , we give a proof of the following.

Theorem 1. For $n \in \mathbb{N}$, let $A: W_{(0)}^{1,2n+2}(\Omega) \rightarrow (W_{(0)}^{1,2n+2}(\Omega))^*$ be defined as above. Then there exist positive constants α, M, C independent of ϵ such that:

- (i) $(Au - Av)(u - v) \geq \alpha |u - v|^{2n+2},$
- (ii) $|Au - Av|^* \leq \{M(\|v\|)^{2n} + 2\epsilon^{n/(n+1)}\} |u - v|,$
- (iii) $|u| \leq C \epsilon^{\frac{1-\mu}{2n+1}}, \quad |u_N| \leq C \epsilon^{\frac{1-\mu}{2n+1}}.$

Proof: (i) In Ciarlet [7], e.g., it is shown that there exists α :

$$\frac{(|x|^{2n} - |y|^{2n}) \cdot (x - y)}{|x - y|^{2n+2}} \geq \alpha, \quad \forall x, y \in \mathbb{R}^2, x \neq y,$$

which yields (i) immediately. (ii) Also, in Ciarlet [7] is given that there exists M' :

$$\frac{||x|^{2n}x - |y|^{2n}y|}{|x - y|(|x| + |y|)^{2n}} \leq M', \quad \forall x, y \in \mathbb{R}^2, x \neq y,$$

which yields that

$$\begin{aligned}
|Au - Av|^* &= \sup_{|w| \leq 1} |(Au - Av)(w)| \\
&\leq \sup_{|w| \leq 1} \left\{ \int_{\Omega} M' |D_{\epsilon} u - D_{\epsilon} v| (|D_{\epsilon} u| + |D_{\epsilon} v|)^{2n} |D_{\epsilon} w| \epsilon d\xi \, d\eta \right. \\
&\quad \left. + \int_{\Omega} |D_{\epsilon} u - D_{\epsilon} v| |D_{\epsilon} w| \epsilon d\xi \, d\eta \right\} \\
&\leq \sup_{|w| \leq 1} \left\{ M |u - v| |w| (|u| + |v|)^{2n} + |u - v| |w| 2\epsilon^{\frac{n}{n+1}} \right\},
\end{aligned}$$

using Hölder's inequality twice ($p_1 = 2n+2$, $p_1' = 2n+2/2n+1$; $p_2 = 2n+1$, $p_2' = 2n+1/2n$) and the fact $(x+y)^{2n} \leq 2^{2n-1} (x^{2n} + y^{2n})$ for $x, y > 0$, see Korovkin [18] p. 25. Thus (ii) holds. For (iii),

$$|u|^{2n+2} \leq Au(u) = G(u) \leq |G|^* |u|,$$

with u replaceable with u_N .

$$\begin{aligned}
|G| &= \sup_{|w| \leq 1} \left| \int_0^1 \epsilon^{1-u} \beta(w(\cdot, 1) + w(\cdot, -1)) d\xi \right| \\
&\leq C \epsilon^{1-\mu - \frac{1}{2n+2}},
\end{aligned}$$

since the trace operator maps $W_{(0)}^{1, 2n+2}(\Omega)$ continuously into $L^{2n+2}(\Gamma)$ with a norm containing a factor $\epsilon^{-\frac{1}{2n+2}}$; (iii) follows.

////

Combining inequalities (i), (ii), and (iii) in Thm. 1 we finally arrive at the following pair of inequalities in (10)

$$(10) \quad |u - u_N| \leq C \varepsilon^{\frac{2n(1-\mu - \frac{1}{2n+2})}{(2n+1)^2}} \inf_{v_N \in V_N} |u - v_N|^{\frac{1}{2n+1}}, \text{ for } \mu > 0,$$

$$|u - u_N| \leq C \varepsilon^{\frac{n}{(2n+1)(n+1)}} \inf_{v_N \in V_N} |u - v_N|^{\frac{1}{2n+1}}, \text{ for } \mu \leq 0$$

where C does not depend on ε .

Proof of (10): To arrive at the ε -dependence we present a standard argument:

$$\begin{aligned} \alpha |u - u_N|^{2n+2} &\leq (Au - Au_N)(u - u_N) \\ &= (Au - Au_N)(u - v_N) \\ &\leq |Au - Au_N|^* |u - v_N| \\ &\leq \{M(|u| + |u_N|)^{2n} + 2\varepsilon^{\frac{n}{n+1}}\} |u - u_N| |u - v_N|; \end{aligned}$$

with $|u|, |u_N| \leq C \varepsilon^{\frac{1-\mu}{2n+1}}$, the result follows.

Due to this reduction in power in Cea's lemma (by $2n+1$) it becomes of interest to obtain a quasioptimal bound in $W^{1,2}$ under further assumptions. This was done by Kosler [19]. We present the argument for our case. The final estimate is (13).

For $v \in W_{(0)}^{1,2}(\Omega)$, let $\|v\|_2 = (\int_{\Omega} |D_{\varepsilon} v|^2 \varepsilon d\xi d\eta)^{1/2}$. Note, that trivially

$$(11) \quad (Aw_1 - Aw_2)(w_1 - w_2) \geq \|w_1 - w_2\|_2^2, \quad \forall w_1, w_2 \in W_{(0)}^{1,2n+2}(\Omega).$$

////

If we now invoke the Mean Value Theorem we get,

$$\forall w_1, w_2 \in W_{(0)}^{1,2n+2} \cap W^{1,\infty}, v \in W_{(0)}^{1,2n+2}$$

$$\begin{aligned} |(Aw_1 - Aw_2)(v)| &= \left| \int_{\Omega} [F(|D_{\epsilon} w_1|^2) D_{\epsilon} w_1 - F(|D_{\epsilon} w_2|^2) D_{\epsilon} w_2] D_{\epsilon} v \, d\xi \, d\eta \right| \\ (12) \quad &= \left| \int_{\Omega} D(F(|\theta|^2)) (D_{\epsilon} w_1 - D_{\epsilon} w_2) D_{\epsilon} v \, d\xi \, d\eta \right| \\ &\leq C (|D_{\epsilon} w_1|_{\infty} + |D_{\epsilon} w_2|_{\infty})^{2n} |w_1 - w_2|_2 |v|_2, \end{aligned}$$

where we have bounded the Hessian (of $\mathbb{R}^2 \ni \theta \rightarrow F(|\theta|^2)$ $\theta \in \mathbb{R}^2$) at some intermediate point: $C(|D_{\epsilon} \theta|_{\infty})^{2n} \leq C(|D_{\epsilon} w_1|_{\infty} + |D_{\epsilon} w_2|_{\infty})^{2n}$. Combining (11) and (12), we get

$$\begin{aligned} \|u_N - v_N\|_2^2 &\leq (Au_N - Av_N)(u_N - v_N) \\ &= (Au - Av_N)(u_N - v_N) \\ &\leq C(|D_{\epsilon} u|_{\infty} + |D_{\epsilon} v_N|_{\infty})^{2n} \|u - v_N\|_2 \|u_N - v_N\|_2, \end{aligned}$$

which with the triangle inequality establishes the following estimate

$$(13) \quad \|u - u_N\|_2 \leq C(|D_{\epsilon} u|_{\infty}, |D_{\epsilon} v_N|_{\infty}) \|u - v_N\|_2,$$

for any $v_N \in V_N \cap W^{1,\infty}$ provided $u \in W_{(0)}^{1,2n+2} \cap W^{1,\infty}$.

We hereby close our remarks about properties of Galerkin solutions and turn our attention to the selection of the basis- or Ansatz functions.

4. Asymptotic expansion (for the selection of basis functions).

Let u_N be the Galerkin solution in V_N (given by (8)) of the variational problem in (9). In the next section, we shall select the basis - or Ansatz functions $\{\psi_j\}_{j=0}^{\infty}$ such as to establish an optimal order of convergence of u_N to u (both dependent upon ϵ) as ϵ tends to zero. To this end we employ in the present section the method of asymptotic expansion and formal matching to "solve" the scaled version of (4). That is, expand u as follows

$$(14) \quad u(\xi, \eta) = \sum_{i,j=0}^{\infty} u^{(i,j)}(\xi, \eta) \epsilon^{p(i,j)}$$

for $\{u^{(i,j)}\}_{i,j}$, and p to be determined. These results will then be applied in the next section in settling said convergence as ϵ tends to zero. It will turn out that the basis functions to be selected are the ones predicted (from the form of $u^{(i,j)}$) by the asymptotic expansions. Recall that formally, (9) and (4) are equivalent by an application of the divergence theorem.

Lemma 1. The initial exponent and term in the expansion (14) are:

(i) For $\mu < 0$:

$$p(0,0) = -\mu, \quad u^{(0,0)}(\xi, \eta) = C_0(\xi) \text{ satisfying}$$

$$C_0'' = -\beta, \quad C_0(0) = C_0(1) = 0;$$

(ii) For $\mu = 0$:

$$p(0,0) = 0, \quad u^{(0,0)}(\xi, \eta) = C_0(\xi) \text{ satisfying}$$

$$C_0'' + ((C_0')^{2n+1})' = -\beta, \quad C_0(0) = C_0(1) = 0;$$

(iii) For $\mu > 0$:

$$p(0,0) = -\frac{\mu}{2n+1}, \quad u^{(0,0)}(\xi, \eta) = C_0(\xi) \text{ satisfying}$$

$$((C_0')^{2n+1})' = -\beta, \quad C_0(0) = C_0(1) = 0.$$

Also, the first η -dependent term has exponent $2 + p(0,0)$ in all three cases

(i) - (iii).

Proof: Let $v = u^{(0,0)}\epsilon^a$ ($a = p(0,0)$). Substitute v for u in the scaled version of (4):

$$\begin{aligned} & D_\epsilon (1 + |D_\epsilon v|^{2n}) D_\epsilon v \\ &= u_{\eta\eta}^{(0,0)} \epsilon^{a-2} + (u_{\eta}^{(0,0)})_{\eta}^{2n+1} \epsilon^{(2n+1)(a-1)-1} \\ &0 + \text{higher order terms} \end{aligned}$$

with boundary conditions at $\eta = \pm 1$:

$$u_{\eta}^{(0,0)} \epsilon^{a-1} + (u_{\eta}^{(0,0)})_{\eta}^{2n+1} \epsilon^{(2n+1)(a-1)} = \pm \beta(\xi) \epsilon^{1-\mu} + \text{h.o.t.}$$

Distinguishing between whether $a < 1$, $a = 1$, or $a > 1$, one learns in either case

from the differential equation that $u_{\eta}^{(0,0)}$ does not vary with η , and then

from the boundary conditions, $u_{\eta}^{(0,0)} = 0$, such that $u^{(0,0)}(\xi, \eta) = C_0(\xi)$.

To conclude the remainder, let the first η -dependent term be called $u^{(i,j)}$ and

set $v = C_0 \epsilon^a + u^{(i,j)} \epsilon^b$, $p(0,0) = a < b = p(i,j)$ and substitute once more for

u :

$$\begin{aligned} & D_\epsilon (1 + (C_0')^2 \epsilon^{2a} + 2C_0' u_{\xi}^{(i,j)} \epsilon^{a+b} + (u_{\xi}^{(i,j)})^2 \epsilon^{2b} + (u_{\eta}^{(i,j)})^2 \epsilon^{2b-2} \eta) \\ & \times (C_0' \epsilon^a + u_{\xi}^{(i,j)} \epsilon^b, u_{\eta}^{(i,j)} \epsilon^{b-1}) \end{aligned}$$

$= 0 + \text{h.o.t.}$

with the boundary conditions at $\eta = \pm 1$

$$(1 + |D_\epsilon v|^{2n}) u_\eta^{(i,j)} = \pm \beta(\xi) \epsilon^{2-b-\mu} + \text{h.o.t.}$$

Distinguishing between whether $a < 0$, $a = 0$, or $a > 0$, one learns after lengthy calculations (formal matching) in each case that:

if $a < 0$: then $\mu = -(2n+1)a > 0$, $b = 2 + a$, and

$$((C_0')^{2n+1})' = -\beta \text{ with } C_0(0) = C_0(1) = 0;$$

if $a = 0$: then $\mu = 0$, $b = 2$, and

$$C_0'' + ((C_0')^{2n+1})' = -\beta \text{ with } C_0(0) = C_0(1) = 0;$$

if $a > 0$: then $\mu = -a < 0$, $b = 2 + a$, and

$$C_0'' = -\beta \text{ with } C_0(0) = C_0(1) = 0.$$

Now, distinguishing between $a < 0$, $a = 0$, and $a > 0$ in each of the cases (i), (ii), and (iii), the proof is completed.

////

Theorem 2. Let $\mu < 0$ and β satisfy

$$\beta \in C^\infty(0,1) \text{ and } \forall k \in \mathbb{N}, \quad \beta^{(2k)}(0) = \beta^{(2k)}(1) = 0.$$

Then the exponents in (14) of the asymptotic expansion for u are given by

$$(15) \quad p(i,j) = 2i - (2j + 1)\mu, \quad \forall i, j \in \mathbb{N};$$

the basis functions are polynomials in η of even degree for fixed ξ , and are determined by the recursive formulae (17) and (18).

Proof: Define $[i,j] = 2i - 2j\mu = p(i,j) + \mu$, $i,j \in \mathbb{N}$; we concatenate terms in (14) according to the following rule (equivalence relation):

$$u^{(i_1, j_1)} - u^{(i_2, j_2)} \Leftrightarrow [i_1, j_1] = [i_2, j_2]$$

$$\Leftrightarrow \begin{cases} i_1 = i_2 & \text{if } j_1 = j_2 \\ \frac{i_1 - i_2}{j_1 - j_2} = \mu & \text{if } j_1 \neq j_2 \end{cases}$$

If we differentiate through in the scaled version of (4), we get:

$$(16) \quad \begin{aligned} & [2F'(T)u_\xi^2 + F(T)]u_{\xi\xi} + \left[\frac{4}{\epsilon^2} F'(T)u_\xi u_\eta \right] u_{\xi\eta} \\ & + \left[\frac{2}{\epsilon^4} F'(T)u_\eta^2 + \frac{1}{\epsilon^2} F(T) \right] u_{\eta\eta} \\ & = 0, \end{aligned}$$

where T denotes $|D_\epsilon u|^2 = u_\xi^2 + \epsilon^{-2}u_\eta^2$ and F is defined in (2).

Define

$$v_1^{(i,j)} = \sum_{[i_1+i_2, j_1+j_2] \leq [i,j]} u_\xi^{(i_1, j_1)} u_\xi^{(i_2, j_2)}$$

$$v_0^{(i,j)} = \sum_{\substack{[i_1+i_2, j_1+j_2] \leq [i,j] \\ [i_2, j_2] \leq [1,0]}} u_\xi^{(i_1, j_1)} u_\eta^{(i_2, j_2)}$$

$$v_2^{(i,j)} = \sum_{\substack{[i_1+i_2, j_1+j_2] \leq [i,j] \\ [i_1, j_1] \geq 2, [i_2, j_2] \geq 2}} u_\eta^{(i_1, j_1)} u_\eta^{(i_2, j_2)}.$$

With these we express

$$(u_{\xi})^2 = \epsilon^{-2\mu} N_1, \text{ where } N_1 = \sum_{i,j=0}^{\infty} v_1^{(i,j)} \epsilon^{[i,j]},$$

$$u_{\xi} u_{\eta} = \epsilon^{-2\mu} \epsilon^2 O, \text{ where}$$

$$\epsilon^2 O = \sum_{[i,j] \geq 2} \sigma^{(i,j)} \epsilon^{[i,j]} = \epsilon^2 \sum_{[i+1,j] \geq 2} \sigma^{(i+1,j)} \epsilon^{[i,j]},$$

$$(u_{\eta})^2 = \epsilon^{-2\mu} \epsilon^4 N_2, \text{ where}$$

$$\epsilon^4 N_2 = \sum_{[i,j] \geq 4} v_2^{(i,j)} \epsilon^{[i,j]} = \epsilon^4 \sum_{[i+2,j] \geq 4} v_2^{(i+2,j)} \epsilon^{[i,j]}.$$

Substituting these in (16), we obtain

$$\begin{aligned} & \epsilon^{-(2n+1)\mu} \{ [2n(N_1 + \epsilon^2 N_2)^{n-1} N_1 + (N_1 + \epsilon^2 N_2)^n] \times \left[\sum_{i,j=0}^{\infty} u_{\xi\xi}^{(i,j)} \epsilon^{[i,j]} \right] \\ & \quad + [4n(n_1 + \epsilon^2 N_2)^{n-1} O] \times [\epsilon^2 \sum_{[i+1,j] \geq 2} u_{\xi\eta}^{(i+1,j)} \epsilon^{[i,j]}] \\ & \quad + [2n(N_1 + \epsilon^2 N_2)^{n-1} N_2 + \epsilon^{-2}(N_1 + \epsilon^2 N_2)^n] \times [\epsilon^2 \sum_{[i+1,j] \geq 2} u_{\eta\eta}^{(i+1,j)} \epsilon^{[i,j]}] \} \\ & + \epsilon^{-\mu} \{ \left[\sum_{i,j=0}^{\infty} u_{\xi\xi}^{(i,j)} \epsilon^{[i,j]} \right] \\ & \quad + \left[\sum_{[i+1,j] \geq 2} u_{\eta\eta}^{(i+1,j)} \epsilon^{[i,j]} \right] \} \\ & = 0 \end{aligned}$$

where we have used Lemma 1. If we multiply by $\epsilon^{(2n+1)\mu}$ and list terms in

descending order in terms of upper index pair for a given power $[i,j]$, we get

$$(17) \quad \sum_{[i+1,j+n] \geq 2} u_{\eta\eta}^{(i+1,j+n)} \epsilon^{[i,j]} + \sum_{[i,j+n] \geq 0} u_{\xi\xi}^{(i,j+n)} \epsilon^{[i,j]} + R = 0.$$

R is an abbreviation for the first $\{...\}$ in the previous equation and contains as leading term $(C_0')^{2n} u_{\eta\eta}^{(i+1,j)} \epsilon^{[i,j]}$. If we multiply the boundary conditions through by $\epsilon^{(2n+1)\mu-1}$, we get

$$(18) \quad \sum_{[i+1,j+n] \geq 2} u_{\eta}^{(i+1,j+n)} \epsilon^{[i,j]} \\ + [N_1 + \epsilon^{2N_2}]^n \sum_{[i+1,j] \geq 2} u_{\eta}^{(i+1,j)} \epsilon^{[i,j]} \\ = \pm \beta(\xi) \epsilon^{2n\mu} \quad \text{at } \eta = \pm 1.$$

Fix $[i,j]$. Solving the Neumann problem (17) and (18) for $u^{(i+1,j+n)}$ gives rise to a solvability condition at $\eta = \pm 1$ yielding an ordinary differential equation in ξ fixing an additive function in $u^{(i,j+n)}$. It is in this O.D.E. with its 0 Dirichlet boundary conditions that the conditions on β become necessary and sufficient. The necessity is most easily seen via an even extension about $\xi = 0$ and $\xi = 1$. Together with Lemma 1 this determines $u^{(i,j)}$ uniquely, $i,j \in \mathbb{N}$. The $u^{(i,j)}$ are even polynomials in n for fixed ξ , in fact induction using (17)-(18) yields that $u^{(i,j)}(\xi, \cdot)$ is of degree $2N$ for $2N \leq [i,j] < 2N+2$.

////

We remark that Thm. 2 would hold for $\mu = 0$ as well if we merely drop the j -term in $p(i,j)$ and combine the leading term in R and the leading term before C in (17) to $(1 + C_0'^{2n}) u_{\eta\eta}^{(i+1)}$ and similarly in (18). Now for $\mu > 0$.

Theorem 3. Let $\mu > 0$ and let β satisfy

$$\beta \in C^\infty(0,1) \text{ and } \forall k \in \mathbb{N}, \beta^{(2k)}(0) = \beta^{(2k)}(1) = 0.$$

Then the exponents in (14) of the asymptotic expansion for u are given by

$$(19) \quad p(i,j) = 2i + \frac{2j-1}{2n+1} \mu, \quad \forall i,j \in \mathbb{N};$$

the basis functions are even polynomials in η for fixed ξ , and are determined by the recursive formulae (20) and (21).

Proof: The same as the proof of Thm. 2, except that the leading term in (17) now is the one from R , $(C_0')^{2n} u_{\eta\eta}^{(i+1,j)}$ and the two first terms in (17) now have indices $(i+1,j-n)$ and $(i,j-n)$ respectively.

$$\begin{aligned} (20) \quad & \sum_{[i+1,j] \geq 2} (C_0')^{2n} u_{\eta\eta}^{(i+1,j)} \epsilon^{[i,j]} + \sum_{i,j=0}^{\infty} (C_0')^{2n} u_{\xi\xi}^{(i,j)} \epsilon^{[i,j]} \\ & + \bar{R} + \sum_{[i+1,j-n] \geq 2} u_{\eta\eta}^{(i+1,j-n)} \epsilon^{[i,j]} + \sum_{[i,j-n] \geq 0} u_{\xi\xi}^{(i,j-n)} \epsilon^{[i,j]} \\ & = 0, \end{aligned}$$

where \bar{R} is R excluding the two first terms of (14) with $p(i,j)$ given by

(19). Similarly, we get for the boundary conditions at $\eta = \pm 1$:

$$\begin{aligned}
 & [N_1 + \epsilon^2 N_2]^n \sum_{[i+1, j] \geq 2} u_n^{(i+1, j)} \epsilon^{[i, j]} \\
 (21) \quad & + \sum_{[i+1, j-n] \geq 2} u_n^{(i+1, j-n)} \epsilon^{[i, j]} \\
 & = \pm \beta(\xi) .
 \end{aligned}$$

Again we have to concatenate the series according to an equivalence relation:

$$\begin{aligned}
 u^{(i_1, j_1)} - u^{(i_2, j_2)} & \Leftrightarrow [i_1, j_1] = [i_2, j_2] \\
 & \Leftrightarrow \begin{cases} i_1 = i_2 & \text{if } j_1 = j_2 \\ \frac{i_1 - i_2}{j_1 - j_2} = -\frac{\mu}{2n+1} & \text{if } j_1 \neq j_2 \end{cases}
 \end{aligned}$$

Also here, $u^{(i, j)}(\xi, \cdot)$ is a polynomial in η of degree $2N$ for $2N \leq [i, j] < 2N + 2$.

////

We remark that the above method is quite general. It could for instance be generalized to a layered material where F might depend also on η . In such cases we would no longer get even polynomials in η as Ansatz functions, but the same type of recursion formulae would determine $u^{(i, j)}$'s η -dependence and the $p(i, j)$ would remain the same.

5. The selection of basis functions.

We will now apply the results from the last section to the selection of the basis- or Ansatz functions $\{\psi_j\}_{j=0}^{\infty}$. This selection fixes the subspaces V_N as defined in (8), and we let u and u_N be the solutions of (7) and (9) respectively. The aim of this section is to make said selection so as to have optimal order of convergence of $u-u_N$ as ϵ tends to zero at fixed N .

Before we state and prove convergence of u_N to u for ϵ tending to zero in the three asymptotic ranges characterized by the sign of μ , we let D_1 be the operator defined by $D_1 u = (d^2 u / d\xi^2)$ mapping $\text{Dom}(D_1) = W^{2,2n+2}_{(0,1)} \cap W^{1,2n+2}_0(0,1) \rightarrow L^{2n+2}(0,1)$.

Theorem 4 Let $\mu = 0$ and $n \in \mathbb{Z}_+$.

$\forall N \in \mathbb{N}$, $\exists C_N$ independent of ϵ such that, if $\beta \in D(D_1^N)$ then

$$|u - u_N| \leq C_N \epsilon^{\frac{2N+2 - \frac{1}{2n+2}}{(2n+1)^2} + \frac{n}{(n+1)(2n+1)}}.$$

Proof: For convenience, let $n = 1$. Let

$$\bar{u}_N = \sum_{j=0}^N u_{jN} \epsilon^{2j}$$

where $u_{jN}(\xi, \eta) = \sum_{i=0}^j x_i^{jN}(\xi) \psi_i(\eta)$. Then

$$D_{\epsilon} \bar{u}_N = \left(\sum_{j=0}^N (u_{jN})_{\xi} \epsilon^{2j}, \sum_{j=0}^N (u_{jN})_{\eta} \epsilon^{2j-1} \right).$$

On the grounds of denseness of $W_0^{1,4}(0,1) \otimes W^{1,4}(-1,1)$ in $W_{(0)}^{1,4}(\Omega)$ and the fact that $A\bar{u}_N \in (W_{(0)}^{1,4}(\Omega))^*$, we restrict v to be of the form $c \cdot d$, where $c \in W_0^{1,4}(0,1)$ and $d \in W^{1,4}(-1,1)$, for convenience. Then (9) becomes with \bar{u}_N

$$\begin{aligned}
 & \int \int_{\Omega} (1 + |D_{\epsilon} \bar{u}_N|^2) \left(\sum_{j=0}^N (u_{jN})_{\xi} \epsilon^{2j}, \sum_{j=0}^N (u_{jN})_{\eta} \epsilon^{2j-1} \right) \\
 & \quad \times (cd, \epsilon^{-1} cd') d\xi d\eta \\
 (22) \quad & = \int_{\Gamma_{\pm 1}} \beta(\xi) cd d\xi \\
 & = 2 \int_0^1 \beta cd\xi d(1),
 \end{aligned}$$

for any such v which is even in η (we shall let $v = u - \bar{u}_N$, even in η) and where

$$\begin{aligned}
 1 + |D_{\epsilon} \bar{u}_N|^2 &= [(u_{0N})_{\eta}^2] \epsilon^{-2} \\
 &+ [1 + (u_{0N})_{\xi}^2 + 2(u_{0N})_{\eta} (u_{1N})_{\eta}] \\
 &+ [2(u_{0N})_{\xi} (u_{1N})_{\xi} + 2(u_{0N})_{\eta} (u_{2N})_{\eta} + (u_{1N})_{\eta}^2] \epsilon^2 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &+ [2(u_{N-1,N})_{\xi} (u_{NN})_{\xi} + (u_{NN})_{\eta}^2] \epsilon^{4N-2} \\
 &+ [(u_{NN})_{\xi}^2 \epsilon^{4N}] .
 \end{aligned}$$

Note, if \bar{u}_N was known a priori to be smooth, we could apply the divergence theorem and get a second order D.E. with B.C. a la (17) and (18) modified by the remark following Thm. 2. This justifies the choice of powers of ϵ in \bar{u}_N .

The left hand side of (22) has coefficients of ϵ raised to powers up to and including $2N-1$; these coefficients remain unchanged if we increase N . To see this, first notice that the coefficient equation for ϵ^{-4} in (22) reads

$$\int \int_{\Omega} (u_{ON})_{\eta}^3 c'd' d\xi d\eta = 0$$

and yields $(u_{ON})_{\eta} = 0$. Next, the coefficient equation for ϵ^0 reads:

$$\begin{aligned} \int \int_{\Omega} (1 + (u_{ON})_{\xi}^2) ((u_{ON})_{\xi} c'd + (u_{1N})_{\eta} c'd') d\xi d\eta \\ = 2d(1) \int_0^1 \beta c d\xi. \end{aligned}$$

We continue to the coefficient equation for ϵ^{2N-1} :

$$\begin{aligned} \int \int_{\Omega} (1 + (u_{ON})_{\xi}^2) ((u_{N-1,N})_{\xi} c'd + (u_{NN})_{\eta} c'd') \\ + \dots \\ + (\text{the coeff. to } \epsilon^{2N-2} \text{ in } |D_{\epsilon} \bar{u}_N|^2) ((u_{ON})_{\xi} c'd + (u_{1N})_{\eta} c'd') d\xi d\eta \\ = 0. \end{aligned}$$

These equations do not change if we increase N . Furthermore, we can solve them exactly; if we integrate by parts we get the first N equations for the

asymptotic expansion (possible if $\beta \in C^{2N}(0,1)$ with $\beta^{(2k)}(0) = \beta^{(2k)}(1) = 0$ for $0 \leq k \leq N$), see Thm. 2 with subsequent remark. However, the coefficient equation for ϵ^{2N+1} :

$$\begin{aligned} & \int \int_{\Omega} (1 + (u_{ON})_{\xi}^2) ((u_{NN})_{\xi} c'd) + \dots \\ & + (\text{the coeff. to } \epsilon^{2N} \text{ in } |D_{\epsilon} \bar{u}_N|^2) ((u_{ON})_{\xi} c'd + (u_{IN})_{\eta} cd') d\xi d\eta \\ & = 0 \end{aligned}$$

up to the coefficient equation for ϵ^{6N+1} (highest power present):

$$\int \int_{\Omega} (u_{NN})_{\xi}^3 c'd \, d\xi d\eta = 0$$

will change as we include more basis functions. Define \bar{u}_N (modulo an additive function of ξ) to be that exact solution $(\sum_{j=0}^N u_{jN} \epsilon^{2j})$ of the coefficient equations of $\epsilon^{-3}, \epsilon^1, \dots, \epsilon^{2N-1}$. $\{u_{jN}\}_{j=0}^N$ is thus defined iteratively provided $\beta \in D(D_1^{N-1})$, see Thm. 2. Furthermore, if we suppose $\beta \in D(D_1^N)$, the coefficient of ϵ^{2N+1} in $A\bar{u}_N(v)$ can be written as

$$- \int \int_{\Omega} (1 + (u_{ON})_{\xi}^2) ((u_{N+1,N+1})_{\eta} cd') \, d\xi d\eta$$

where we write cd' as $(\frac{1}{\epsilon} v_{\eta}) \epsilon^{\frac{1}{2}} \epsilon^{1 - \frac{1}{2}}$. Thus, $\forall v \in W_{(0)}^{1,4}(\Omega)$:

$$(Au - A\bar{u}_N)(v) = G(v) - A\bar{u}_N(v)$$

$$= G(v) - (G(v) + R(\bar{u}_N, v)) ,$$

where (for ϵ bounded; $|R| \rightarrow \infty$ as ϵ^{6N+1} for $\epsilon \rightarrow \infty$)

$$|R(\bar{u}_N, v)| \leq C_N |v| \epsilon^{2N+2-\frac{1}{4}}$$

which together with strong monotonicity implies

$$\begin{aligned} \alpha \|u - \bar{u}_N\|^4 &\leq |(Au - A\bar{u}_N)(u - \bar{u}_N)| \\ &\leq C_N \|u - \bar{u}_N\| \epsilon^{2N+2-\frac{1}{4}} \end{aligned}$$

With (10) this implies (with a new constant C_N)

$$\|u - u_N\| \leq C_N \epsilon^{\frac{2N+2-\frac{1}{4}}{9} + \frac{1}{6}}$$

for $n=1$. Larger n change only the trivial and tiresome algebra.

////

We immediately obtain the following estimate in $W^{1,2}$:

Corollary 1 Under the same hypotheses as in Theorem 3 and supposing that $u, \bar{u}_N \in W^{1,\infty}$, C_N independent of ϵ such that

$$\|u - u_N\|_2 \leq C_N \epsilon^{2N+3/2} .$$

Proof: Again, let $n=1$. Now string

$$\begin{aligned}
 \|u - \bar{u}_N\|_2^2 &\leq (Au - A\bar{u}_N)(u - \bar{u}_N) \\
 &\leq |\hat{R}(\bar{u}_N, u - \bar{u}_N)| \epsilon^{2N+1}, \\
 \hat{R}(u_N, v) &= \left\{ \int \int_{\Omega} (1 + (u_{0N})_{\xi}^2) - (u_{N+1, N+1})_{\eta} \, cd' \, d\xi d\eta \right. \\
 &\quad + \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad + \epsilon^{4N} \left\{ \int \int_{\Omega} (u_{NN})_{\xi}^3 \, c'd \, d\xi d\eta \right. \\
 &\quad \leq C_N \epsilon^{\frac{1}{2}} \|v\|_2, \quad \forall v = cd \in W_{(0)}^{1,4},
 \end{aligned}$$

inequality (13) for $v_N = \bar{u}_N$ together to obtain the estimate; w.l.o.g. we

have let $v = c \cdot d$, $c \in W_0^{1, 2n+2}(0, 1)$, $d \in W^{1, 2n+2}(-1, 1)$.

////

The next Thm. covers the case $\mu < 0$.

Theorem 5. Let $\mu \in \mathbb{R}_-$, $n \in \mathbb{Z}_+$. $\forall N \in \mathbb{N}, \exists C_N$ independent of ϵ such that if $\beta \in D(D_1^N)$ then

$$\|u - u_N\| \leq C_N \epsilon^{\frac{e(N, \mu)}{(2n+1)^2} + d(n)}$$

where

$$e(N, \mu) = 2N+2-\mu - \frac{1}{2n+2}, \quad \text{and}$$

$$d(n) = \frac{n}{(n+1)(2n+1)} .$$

Proof: Let $([i,j] = 2i - 2j\mu, i, j \in \mathbb{N})$

$$\bar{u}_N = \sum_{\substack{[i,j] < 2N+2 \\ i,j \in \mathbb{N}}} u_N^{(i,j)} \epsilon_{[i,j]-\mu}$$

where $u_N^{(i,j)}(\xi, \eta) = \sum_{k=0}^i x_k^{(i,j)}(\xi) \psi_k(\eta)$, where (in case $\mu \in \mathbb{Q}$) we take the representative (i,j) for which $-j\mu < 1$. The remainder of the proof is then a mere imitation of the procedure shown in the previous proof with the aid (as also there) of Thm. 2.

////

For $\mu > 0$, we need an extra assumption on β :

Let $\beta \in (W^{1,2n+2})^* = W^{-1, \frac{2n+2}{2n+1}}$ satisfy:

The function $c_0 \in W^{1,2n+2}_0(0,1)$ defined as the solution of

$$\int_0^1 \{(c'_0)^{2n+1} c' + \beta c\} d\xi = 0, \quad \forall c \in W^{1,2n+2}_0(0,1),$$

(A) satisfies that there exists $u \in W^{1,2n+2}_{(0)}(\Omega)$, solution of

$$\int_{\Omega} (c'_0)^{2n} u_n v_n = g(v), \quad \forall v \in W^{1,2n+2}_{(0)}(\Omega)$$

for any given $g \in (W^{1,2n+2}_{(0)}(\Omega))^*$.

Theorem 6 Let $\mu \in \mathbb{R}_+$, $n \in \mathbb{Z}_+$. Let β satisfy assumption (A). $\forall N \in \mathbb{N}$,

let $\beta \in D(D_1^N)$. Then $\exists C_N$ independent of ϵ such that

$$|u - u_N| \leq C_N \epsilon \frac{e(N, \mu) + d(n, \mu)}{(2n+1)^2},$$

where, again

$$e(N, \mu) = 2N+2-\mu - \frac{1}{2n+2}, \text{ and}$$

$$d(n, \mu) = 2n(1-\mu - \frac{1}{2n+2})$$

Proof: Let (with the [...]) notation of Thm. 3)

$$\bar{u}_N = \sum_{\substack{[i,j] < 2N+2 \\ i,j \in \mathbb{N}}} u_N^{(i,j)} \epsilon^{[i,j] - \frac{\mu}{2n+1}}$$

where $u_N^{(i,j)} = \sum_{k=0}^i x_k^{(i,j)} \psi_k$, where (for $\mu \in \mathbb{Q}$) we take that representative (i,j) for which $\frac{2j}{2n+1} \mu < 1$. Otherwise the proof is like that of Thm. 4 with the aid of Thm. 3 and the inequality (10) for $\mu > 0$.

////

Assumption (A) is rather restrictive and can be dispensed with through containing ourselves to local estimates or estimates in weighted norms.

As presented in Corollary 1 for $\mu=0$, it is possible also for $\mu \neq 0$ to obtain estimates in the $W_{(0)}^{1,2}$ - norm rather than the $W_{(0)}^{1,2n+2}$ - norm.

Corollary 2 Let $\mu < 0$. Under the same hypotheses as in Thm. 4 and supposing that u and $u_N \in W^{1,\infty}$, $\exists C_N$ independent of ϵ such that

$$\|u - u_N\|_2 \leq C_N \epsilon^{e(N, \mu)}$$

where $e(N, \mu)$ is the one defined in Theorem 4 with $n=0$.

The proof is like that of Corollary 1.

Corollary 3. Let $\mu > 0$. Under the same hypotheses as in Thm. 5 and supposing that $\epsilon^{\frac{\mu}{2n+1}} u$ and $\epsilon^{\frac{\mu}{2n+1}} u_N \in W^{1, \infty}$, C_N independent of ϵ such that

$$\|u - u_N\|_2 \leq C_N \epsilon^{e(N, \mu) - \frac{2n\mu}{2n+1}}$$

where $e(N, \mu)$ is the one defined in Theorem 5 for $n=0$.

Proof: As in Corollary 1,

$$\|u - \bar{u}_N\|_2 \leq C_N \epsilon^{e(N, \mu)},$$

but (13) now states

$$\begin{aligned} \|u - u_N\|_2 &\leq C (\|Du\|_{\infty} + \|D\bar{u}_N\|_{\infty})^{2n} \|u - \bar{u}_N\|_2 \\ &\leq C \epsilon^{-\frac{2n\mu}{2n+1}} \|u - \bar{u}_N\|_2. \end{aligned}$$

The combination of the two inequalities completes the proof.

////

It is important to note that we have made the selection of basis or Ansatz functions $\{\psi_i\}_{i=0}^N$ (and thus V_N) by constructing \bar{u}_N out of $\{u_N^{(i,j)}\}_{[i,j] < 2N+2}$, where $u_N^{(i,j)}(\xi, \eta) = \sum_{k=0}^i x_k^{(i,j)}(\xi) \psi_k(\eta)$ are determined through solving equations which are the weak variants of the formal

matching equations already encountered in Section 4.

We observe that for our model problem (the particular form of F given in (2)), we obtain the same selection of $\{\psi_i\}_{i=1}^N$ (namely the even polynomials) in all three asymptotic ranges ($\mu=0$ meaning finite limit load, $\mu<0$: zero limit load, and $\mu>0$: infinite limit load). This would hold for more general F (see the paragraph following (9)) including such that have a factor depending upon η (e.g., layered materials); here the selection of $\{\psi_i\}_{i=0}^N$ would be made according to the same scheme, only, the sequence of second order differential equations determining ψ_i would have coefficients depending upon η and $\{\psi_i\}_{i=0}^N$ would no longer merely be polynomials.

In the next section we intend to show that this choice is optimal with respect to $W_{(0)}^{1,2}$ - norm convergence under appropriate boundedness conditions on $D_\varepsilon u$, $D_\varepsilon u_N$ in L^∞ .

6. The optimality of above selection of basis functions (at fixed N).

Define for $\{\phi_i\}_{i=0}^N \in W^{1,2n+2}_{(-1,1)}$,

$$W_N = \left\{ \sum_{i=0}^N c_i \phi_i : c_i \in W^{1,2n+2}_0(0,1) \right\},$$

and let $\{\psi_i\}_{i=0}^N$ be the basis functions selected in the previous section (defining V_N). There we established the following convergence rate in $W^{1,2}_{(0)}$ -norm for ϵ tending to zero:

$$\|u - u_N\|_2 \leq C_N \Psi(\epsilon)$$

under certain assumptions (see Corollaries 1, 2, and 3), where

$$\Psi(\epsilon) = \begin{cases} \epsilon^{2N+\frac{3}{2}-\mu} & , \text{ for } \mu \leq 0 \\ \epsilon^{2N+\frac{3}{2}-\mu} - \frac{2n}{2n+1} \mu & , \text{ for } \mu > 0 \end{cases}.$$

Regarding the question of optimality of the selection of $\{\psi_i\}_0^N$ we now state and prove

Theorem 7. Let $\mu \leq 0$. Under the hypotheses of Corollaries 1 and 2, suppose $\{\phi_i\}_{i=0}^N$ exists such that (for all $N \in \mathbb{N}$)

$$\inf_{w_N \in W_N} \|u - w_N\|_2 \leq \phi(\epsilon)$$

for some function ϕ with the property that

$$\frac{\phi(\epsilon)}{\Psi(\epsilon)} \text{ is bounded for all } \epsilon > 0.$$

Then $w_N \approx v_N$.

Proof: The proof is by induction on N . Take the particular function

$\bar{v}_N = \sum_{[i,j] \leq 2N} u_N^{(i,j)} \epsilon^{[i,j]-\mu}$ related to \bar{u}_N of Thm. 4 and 5. $\exists \bar{w}_N \in w_N$ such that

$$\|\bar{v}_N - \bar{w}_N\|_2 \leq \phi(\epsilon).$$

In particular, let $N = 0$:

$$\|\bar{v}_0 - \bar{w}_0\|_2 = \|u_0^{(0,0)} \epsilon^{-\mu} - \bar{w}_0\|_2 = o(\epsilon^{\frac{3}{2} - \mu}).$$

We thus get

$$\|u_0^{(0,0)} - \bar{w}_0 \epsilon^{\mu}\|_2 = o(\epsilon^{\frac{3}{2}})$$

such that (if we denote $w_0^{(0,0)} = \bar{w}_0 \epsilon^{\mu}$)

$$\int_{\Omega} \epsilon |u_0^{(0,0)} - w_0^{(0,0)}|^2 dx \leq \|u_0^{(0,0)} - w_0^{(0,0)}\|^2 \leq C_0 \epsilon^3.$$

whence $w_0^{(0,0)} = u_0^{(0,0)}$ and $\bar{u}_0 = \bar{w}_0 \in w_0$.

Suppose next that $\{\psi_i\}_{i=0}^{N-1} \subseteq \text{span} \{\phi_i\}_{i=0}^N$, then there exists

$\bar{w}_N \in w_N$ such that ($[...]$ is defined in Thm. 2)

$$\begin{aligned} & \|\bar{v}_N - \bar{w}_N\|_2 \\ &= \left\| \sum_{[i,j] \leq 2N} u_N^{(i,j)} \epsilon^{[i,j]-\mu} - \bar{w}_N \right\|_2 \\ &= o(\epsilon^{2N + \frac{3}{2} - \mu}), \end{aligned}$$

where $u_N^{(i,j)} = \sum_{k=0}^i x_k^{(i,j)} \psi_k$ as before. This means that

$$\begin{aligned} & \left\| \sum_{[i,j] \leq 2N} u_N^{(i,j)} \epsilon^{[i,j] - \mu - w_N} \right\|_2 \\ &= \left\| x_N^{(N,0)} \psi_N \epsilon^{2N-\mu} + \left(\sum_{k=0}^{N-1} x_k^{(N,0)} \psi_k \epsilon^{2N-\mu} + \sum_{[i,j] < 2N} u_N^{(i,j)} \epsilon^{[i,j] - \mu - w_N} \right) \right\| \\ &= O(\epsilon^{2N-\mu+\frac{3}{2}}), \end{aligned}$$

such that upon taking the limit $\epsilon \rightarrow 0$, ψ_N span $\{\phi_i\}_{i=0}^N$. This completes the proof.

////

We note that it is possible to strengthen this Thm. to allow ϕ for which

$$\frac{\phi(\epsilon)}{\psi(\epsilon)} \epsilon^p \leq o(1) \quad , \quad \text{as } \epsilon \rightarrow 0 \text{ for any } p \in [0,2],$$

by employing

$$\epsilon^{-\frac{1}{2}} \left(\iint_{\Omega} \left| \frac{\partial}{\partial \eta} (\bar{v}_N - \bar{w}_N) \right|^2 \right)^{\frac{1}{2}} \leq \|\bar{v}_N - \bar{w}_N\|_2 = o(\epsilon^{2N-\mu+\frac{3}{2}p}).$$

This means that even if we were satisfied with a convergence rate of $2N+\frac{3}{2} - \mu - p$, $p \in [0,2]$, we would still need to use $\{\psi_i\}_{i=0}^N$. We shall employ this technique in the next theorem.

Theorem 8. Let $\mu > 0$. Under the hypotheses of Corollary 3, suppose

$\{\phi_i\}_{i=0}^N$ exists such that (for all $N \in \mathbb{N}$)

$$\inf_{w_N \in W_N} \|u - w_N\|_2 \leq \phi(\epsilon)$$

for some function ϕ with the property that

$$\frac{\phi(\epsilon)}{\psi(\epsilon)} \epsilon^{p + \frac{4n+1}{2n+1} \mu} = o(1), \text{ as } \epsilon \rightarrow 0, \text{ for some } p + \frac{4n+1}{2n+1} \mu \in [0, 2].$$

Then $W_N \subset V_N$.

Proof: For $p + \frac{4n+1}{2n+1} \mu \in [0, \frac{3}{2}]$, the proof would merely copy that of the

previous Thm. with $\bar{v}_N = \sum_{[i,j] \leq 2N} u_N^{(i,j)} \epsilon^{[i,j] - \frac{\mu}{2n+1}}$ and

$$[i,j] = 2i + \frac{2j\mu}{2n+1}.$$

As is, let $N = 0$:

$$\begin{aligned} \|\bar{v}_0 - \bar{w}_0\|_2 &= \|u_0^{(0,0)} \epsilon^{-\frac{\mu}{2n+1}} - \bar{w}_0\| = o\left(\epsilon^{-\mu - \frac{2n}{2n+1} \mu - p + \frac{3}{2}}\right) \\ &= o\left(\epsilon^{-\frac{4n+1}{2n+1} \mu - p + \frac{3}{2}}\right), \end{aligned}$$

and

$$\epsilon^{-\frac{1}{2}} \left(\iint_{\Omega} \left| \frac{\partial}{\partial n} (u_0^{(0,0)} \epsilon^{-\frac{\mu}{2n+1}} - \bar{w}_0) \right|^2 \right)^{\frac{1}{2}} = o\left(\epsilon^{\frac{4n+1}{2n+1} \mu + \frac{3}{2} - p}\right).$$

Since $\frac{\partial}{\partial n} u_0^{(0,0)} = 0$, letting $\epsilon \rightarrow 0$, $\phi_0' = 0$, and $\psi_0 \in \text{span}\{\phi_0\}$.

With this modification, the induction step will carry over from Thm. 7.

////

This ensures optimality for $n=0$ and otherwise $\mu < 2(2n+1)/(4n+1)$.

We emphasize that for $\mu > 0$ we used an L^∞ - bound on the gradients of $\epsilon^{-\frac{\mu}{2n+1}} u$ and $\epsilon^{-\frac{\mu}{2n+1}} u_N$, which implies that we have an ϵ -dependent constant $C(|Du|_\infty, |Dv|_\infty)$ in (13).

In Thms. 4-6 we established convergence rates for ϵ tending to zero with N fixed. Even though we are dealing with "thin" structures, ϵ nevertheless has some finite value over which we may have no control. Even if ϵ is "very small" we have seen that convergence rates are limited by the degree of smoothness of the load β . It is therefore of interest and importance to prove convergence for the number of reduced models (N) increasing.

7. Optimal convergence for increasing number of reduced models.

Let $\epsilon > 0$ be fixed. Let u_N be the Galerkin solution in V_N (given in (8)) of (9) as before. Proving convergence of $u - u_N$ for N tending towards infinity reduces to an approximation theoretical problem because of (10). For completeness we outline it here, cf. [31], Babuška, Szabo, and Katz [3], and Canuto and Quarteroni [5] for L^2 -theory and Meinardus [23] and Jackson [13] for L^∞ -theory. We then interpolate in order to obtain L^{2n+2} -estimates. Details are found in [14].

The solution u is an even function in η so we define for any $p \in [2, \infty]$, $m \in \mathbb{Z}_+$:

$$\hat{W}^{m,p}(-1,1) = W^{m,p}(-1,1) \cap \{v : v \text{ even}\}.$$

Let $\hat{W}_{(\xi,\eta)}^{(1,m)p} = \hat{W}^{1,p} \cap \{v \in \hat{W}_\eta^{m,p} : v_\xi \in \hat{W}_\eta^{m,p}\}$, $2 \leq p \leq \infty$, $m \in \mathbb{Z}_+$, where $v \in \hat{W}_\eta^{m,p}$ if v and its first m partials w.r.t. η belong to L^p in the usual generalized sense. Define

$$|v|_{\hat{W}_{(\xi,\eta)}^{(1,m)p}} = |v|_{W^{1,p}} + \|\partial^{m+1} v / \partial \xi \partial \eta^m\|_{L^p} + \|\partial^m v / \partial \eta^m\|_{L^p}.$$

Let P_N be the space of functions which are polynomials in η of degree at most N and \hat{P}_N be those functions in P_N which are even in η .

Using fairly standard argument along the lines of the references cited above, one obtains:

Theorem 9. For all $N \in \mathbb{N}$, $m \in \mathbb{Z}_+$, there exists an operator

$$\mathcal{L}_{2N} : \hat{W}_{(\xi,\eta)0}^{(1,m)2} \rightarrow \hat{P}_{2N} \subseteq \hat{W}^{1,\infty}(\Omega) \subseteq \hat{W}^{1,2}(\Omega)$$

which is a continuous linear mapping:

$$I - \mathcal{L}_{2N} \in \mathcal{L}(\hat{W}_{(\xi, n)}^{(1, m)2}, \hat{W}^{1, 2}(\Omega)), \text{ and}$$

$$I - \mathcal{L}_{2N} \in \mathcal{L}(\hat{W}_{(\xi, n)}^{(1, m)\infty}, \hat{W}^{1, \infty}(\Omega)),$$

with norms bounded by a constant times $(2N)^{1-m}$.

We then obtain the main result of this section.

Theorem 10. Let $u \in \hat{W}_{(\xi, n)}^{(1, m)2n+2}(\Omega)$ for an $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}_+$. Then there exists a positive C independent of u and N such that for $N \in \mathbb{Z}_+$

$$\inf_{v \in V_N} \|u - v\|_{\hat{W}^{1, 2n+2}} \leq C(2N)^{1-m} \|u\|_{\hat{W}^{(1, m)2n+2}}.$$

Proof. Employ interpolation by the K-method, see Bergh & Lofstrom [4] Thm.

6.45 (5), we obtain from Thm. 9 (choose $\theta = 1/(n+1)$)

$$(I - \mathcal{L}_{2N}) \in \mathcal{L}(\hat{W}_{(\xi, n)}^{(1, m)2n+2}, \hat{W}^{1, 2n+2}(\Omega))$$

with a bound on the norm proportional to $(2N)^{1-m}$.

////

This yields the immediate

Corollary 4. Let $u \in \mathbb{R}$, $n \in \mathbb{Z}_+$. $\forall N \in \mathbb{Z}_+$, $\exists C > 0$ independent of N

such that, if $u \in \hat{W}_{(\xi, n)}^{(1, m)2n+2}(\Omega)$,

$$\|u - u_{2N}\| \leq C (2N)^{\frac{1-m}{2n+1}},$$

u being the exact Galerkin solution of (7).

Proof: Combine (10) with Theorem 10.

////

The similar Corollary about $W^{1,2}$ -rates out of Thm. 9 is

Corollary 5. Let $\mu \in \mathbb{R}$, $n \in \mathbb{Z}_+$. $\forall N \in \mathbb{Z}_+$, $\exists C > 0$ independent of N such that, if $u \in \hat{W}_{(\xi, \eta)}^{(1, m)2}(\Omega) \cap W^{1, \infty}$, $\{u_{2N}\}$ bounded in $W^{1, \infty}$ uniformly,

$$\|u - u_N\|_2 \leq C(2N)^{1-m}.$$

Proof: Combine (13) with Theorem 9.

////

We will be in the position to show that the subspace V_N gives quasi-optimal rates of convergence in $W^{1,2}$ (in the sense of n -widths) after the following definitions (i.e., in general one could only improve the constant not the rate):

The (Kolmogorov) N -width of W (a subset of $W^{1,2}$) in $W^{1,2}(\Omega)$ is given by

$$d_N(W; W^{1,2}) = \inf \{E(W; W_N) : W_N \text{ is an } N\text{-dimensional subspace of } W\},$$

where

$$E(W; W_N) = \sup_{x \in W} \inf_{y \in W_N} \|x - y\|$$

is the deviation of W from W_N .

A subspace W_N of W of dimension at most N for which

$$d_N(W; W^{1,2}) = E(W; W_N)$$

is called an optimal subspace for $d_N(W, W^{1,2})$. The notation follows that of the monograph by Pinkus [27] .

Let $\hat{W} = W^{m,2n+2}(-1,1)$. In said monograph [27], the N -width is computed in Chapter VII. Set $B = \{v \in W: |v| \leq 1\}$, then there exist positive constants C and D independent of N for which

$$D(2N)^{-m} \leq d_N(B; L^2) \leq C(2N)^{-m} .$$

This yields immediately

$$D(2N)^{1-m} \leq d_N(B, W^{1,2}(-1,1)) \leq C(2N)^{1-m} .$$

In view of the denseness of $W^{1,2n+2}_0(0,1) \oplus W^{1,2n+2}(-1,1)$ in $W^{1,2n+2}_0(\Omega)$, this shows that the estimate in Corollary 5 is optimal (modulo the constant C) and that this is obtained by the subspace V_N . Thus $\{\psi_i\}_{i=0}^N$ (or any $\{\phi_i\}_{i=0}^N$ with the same span) provides a quasioptimal choice with regard to convergence at fixed ϵ for the number of reduced models (N) increasing.

8. Computational results

Consider the numerical problem of finding the dimensionally reduced solution of (7) within V_N . Let $n=1$ in (2) as an example. We will consider two loads β : $\beta=1$ and $\beta(x) = \lambda\pi^2(1+3\lambda^2\cos^2\pi x)\sin\pi x$. We limit ourselves in advance to $N = 0, 1$, and 2 .

Select $\phi_0=1$, $\phi_2=\eta^2$, and $\phi_4=\eta^4$. These form the same span as the first three ψ_j as could be found from Thms. 1 and 2. Let

$$u_N = \sum_{j=0}^N v_{2j}(\xi)\phi_{2j}(\eta)$$

with $v_{2j} \in W_0^{1,4}(0,1)$. Substituting this expression into (7) yields a system of N nonlinear, second order ordinary differential equations in ξ , upon integration w.r.t. η over $[-1,1]$, to be solved for $\{v_{2j}\}_{j=0}^N$. We convert this into a first order system. Let $w_{j+1} = v_{2j}$, and $w_{j+N+2} = v'_{2j}$ for $0 \leq j \leq N$. We arrive at

$$A(\underline{w})\underline{w}' = R(\underline{w}, \beta),$$

where $\underline{w} = (w(j))_{j=1}^{2N+2}$, $A \in \mathbb{R}^{2N+2 \times 2N+2}$, and $R \in \mathbb{R}^{2N+2}$. Upon inversion of A , the system is in a state such that it is numerically solvable using the nonlinear ODE-system's solver written by V. Majer, cf. Babuska and Majer [21], and Majer [22].

From that point on we have taken mainly two approaches. The first to be mentioned is applicable in general for nonsmooth or noncompatible boundary loads β as well as for all three asymptotic ranges ($\mu < 0$, $\mu = 0$, and $\mu > 0$). The second is applicable only for smooth, compatible β and for $\mu \leq 0$, in which cases an asymptotic expansion can be generated automatically.

We now describe the performance of the first approach. Let $\beta=1$ (non-compatible) as an example.

The exact energy (and solution) being unavailable, it is rather found by extrapolation from the energies $E(u_0)$, $E(u_1)$, and $E(u_2)$.

It is intuitively obvious that in the range $|D_\epsilon u| \ll 1$, $E(u_N) - E(u)$ can be interpreted as the square of the difference in the H^1 -norm. Also, there is some relation to the difference in the $W^{1,4}$ -norm. Define

$$\phi: \mathbb{R}^2 \ni p \rightarrow \phi(p) = (1 + \frac{1}{2}|p|^2)|p|^2. \quad \text{Then}$$

$$E(u_N) - E(u) = \frac{1}{2} \iint_{\Omega} \int_0^1 d_p \phi(D_\epsilon u + s(D_\epsilon(u_N - u))) \cdot D_\epsilon(u_N - u) ds d\xi d\eta - G(u_N - u)$$

and since

$$\frac{1}{2} \iint_{\Omega} \epsilon d_p \phi(D_\epsilon u) \cdot D_\epsilon(u_N - u) - G(u_N - u) = 0$$

(u is the minimizer) we get

$$E(u_N) - E(u) = \frac{1}{2} \iint_{\Omega} \epsilon D_\epsilon(u_N - u) \underline{H} D_\epsilon(u_N - u),$$

$$\text{where } \underline{H} = \int_0^1 \int_0^s d_p^2 \phi(D_\epsilon u + t D_\epsilon(u_N - u)) dt ds \quad \text{with}$$

$$(d_p^2 \phi)_{ij} = \delta_{ij}(1 + |p|^2) + 2p_i p_j. \quad \text{Thus we can estimate with}$$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{41}}{24} > 0,$$

$$E(u_N) - E(u) \geq 4 \lambda_1 \epsilon \iint_{\Omega} \frac{1}{2} |D_\epsilon(u_N - u)|^2 + \frac{1}{4} |D_\epsilon(u_N - u)|^4, \quad \text{and}$$

$$E(u_N) - E(u) \leq \epsilon \iint_{\Omega} \frac{1}{2} |D_\epsilon(u_N - u)|^2 + 3 \lambda_2 \{ |D_\epsilon u|^2 |D_\epsilon(u_N - u)|^2 + |D_\epsilon(u_N - u)|^4 \}.$$

If we furthermore assume that $u_N, u \in W^{1,\infty}$ with $\|D_\epsilon u_N\|_\infty, \|D_\epsilon u\|_\infty < M$ for all N for some positive constant M , then said relationships between the difference in energies and the norms of the energies emerge:

$$E(u_N) - E(u) \sim \|u_N - u\|_{H^1}^2$$

$$\lambda_1 \|u_N - u\|_{H^1}^4 \leq E(u_N) - E(u) \leq \left(\frac{1}{2} + 15\lambda_2 M\right)^2 |\Omega|^{\frac{1}{2}} \|u_N - u\|_{H^1}^2.$$

The latter assumption is dubious however in the limit $\epsilon \rightarrow 0$ for $\mu > 0$ (see Lemma 1). With said assumption, Kosler [19] proved the usual Cea's Lemma type estimate $\|u - u_N\|_{H^1} \leq C_N \inf_{v_N \in V_N} \|u - v_N\|_{H^1}$. See (13). For $\mu \leq 0$, we thus expect "linear" behavior (as well as for $\mu = 0$, if $M \ll 1$). See Corollaries 1 and 2.

The expected rates of convergence (for $\epsilon \rightarrow 0$) measured w.r.t. difference in energy are (cf. Vogelius & Babuska [30] for $\mu=1$, linear case)

| | Linear ($N=0,1,2$) | Nonlinear* ($N=0,1,2$) |
|------------|-------------------------|---|
| $\mu = -1$ | (5,6,6) | $(\frac{17}{18}, \frac{19}{18}, \frac{19}{18})$ |
| $\mu = 0$ | (3,4,4) | $(\frac{13}{18}, \frac{5}{6}, \frac{5}{6})$ |
| $\mu = 1$ | (1,2,2) | $(\frac{1}{9}, \frac{1}{3}, \frac{1}{3})$ |

where the * indicates that they are based upon $W^{1,4}$ -estimates. If we used the $W^{1,2}$ -estimates, we would get the rates for the linear case except that for $\mu=1$, we get $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ instead.

| μ | ϵ | $E(u_0)-E(u)$ | $E(u_1)-E(u)$ | $E(u_2)-E(u)$ |
|-------|-------------|---------------|---------------|---------------|
| -1 | .1 | 6.35 e-8 | 2.93 e-10 | 1.35 e-12 |
| | .01 | 6.64 e-13 | 2.80 e-16 | 1.18 e-19 |
| | .001 | 6.66 e-18 | 0 | 0 |
| | .0001 | 7.00 e-23 | 0 | 0 |
| 0 | .1 | 6.31 e-6 | 2.92 e-8 | 1.34 e-10 |
| | $10^{-3/2}$ | 2.07 e-7 | 2.92 e-10 | 4.12 e-13 |
| | .01 | 6.62 e-9 | 3.90 e-12 | 2.29 e-15 |
| | $10^{-5/2}$ | 2.10 e-10 | 0 | 0 |
| +1 | 1. | 5.90 e-9 | 6.44 e-10 | 7.03 e-11 |
| | .1 | 9.92 e-10 | 4.58 e-12 | 2.11 e-14 |
| | .01 | 1.04 e-10 | 4.40 e-14 | 1.87 e-17 |
| | .001 | 1.03 e-11 | 1.00 e-14 | 9.69 e-18 |

Table 1

Differences in energy dependent upon ϵ and μ for $\beta = 1$.

The last column $E(u_2)-E(u)$ has an extra factor ϵ built in on account of the extrapolation. We notice the expected linear rates for $\mu < 0$, to some extent $\mu = 0$, and in an ϵ -range for $\mu > 0$, whereas there is an expected marked slowdown for $\mu > 0$ as $\epsilon \rightarrow 0$.

The convergence rates for $N \rightarrow \infty$ exhibit the same type of behavior. From Vogelius and Babuška [31], we expect $E(u_N)-E(u) \sim N^{-4}$, see Figs. 1, 2, and 3.

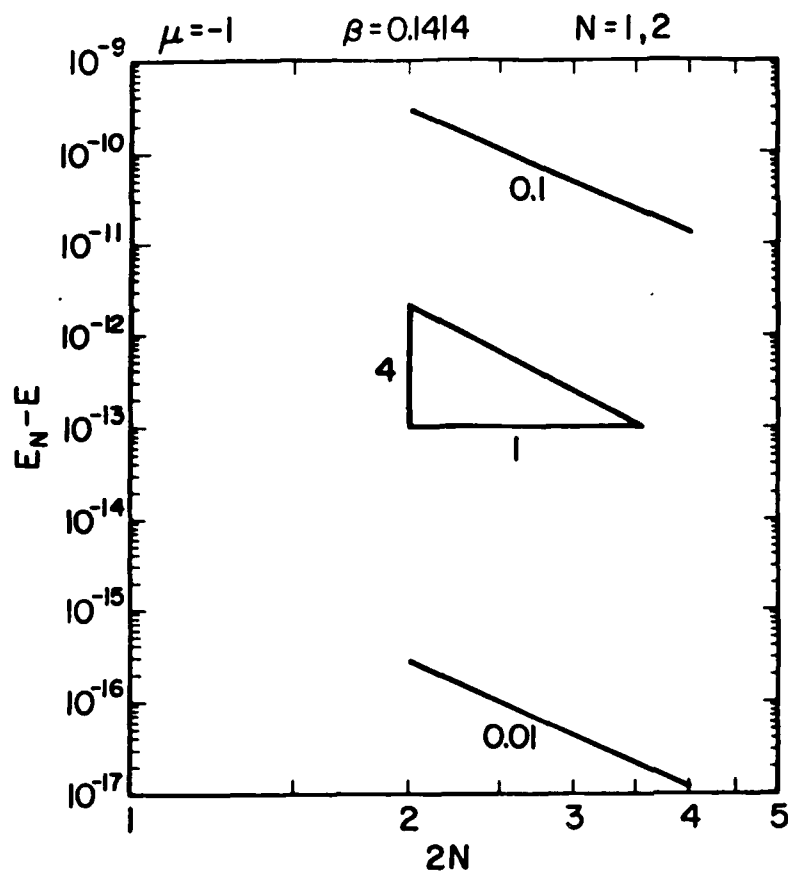


Fig. 1. Error in energy
vs. degree of polynomial
for $\mu = -1$ at $\epsilon = .1$
and $.01$.

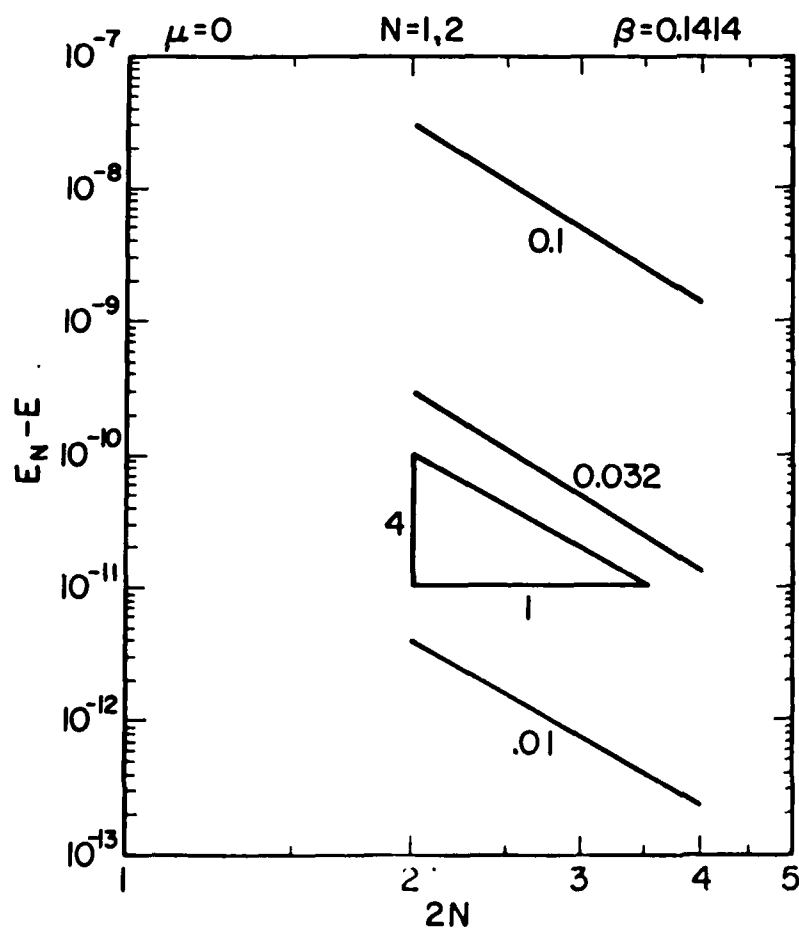


Fig. 2. Error in energy
vs. degree of polynomial
for $\mu = 0$ at $\epsilon = .1$,
 $10^{-3/2}$, and $.01$.

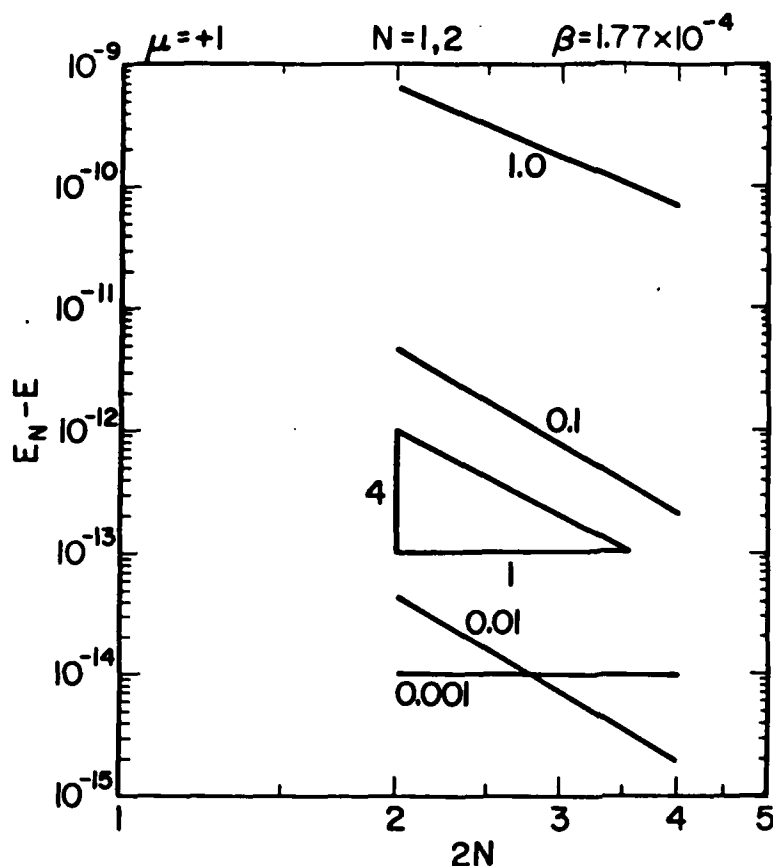


Fig. 3. Error in energy vs. degree of polynomial for $\mu = +1$ at $\epsilon = 1., .1, .01$, and $.001$.

The higher order (N) models become necessary not only for ϵ not small, but also when sufficient smoothness is absent (boundary- and interior layers). Pertaining to the latter case, we also need to show extra consideration when we want to resolve the (from a practical viewpoint very interesting) "nonsmoothness" in the solution well. This would namely mean that we would like to resolve w_j , j large, well. But w_j 's support lies in a (still) narrow(er) band near the singularity (for j/ϵ large enough) calling for separate adaptively generated meshes for those w_j . For the linear case, see Vogelius and Babuška [32]. See Figs. 4, 5, 6, 7, 8, and 9.

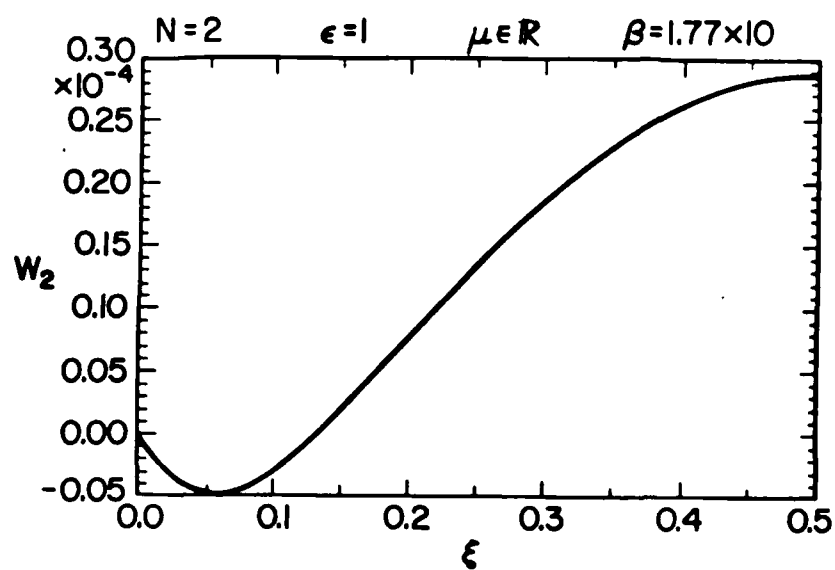


Fig. 4. Coefficient function w_2 for $\epsilon=1$. ($\mu \in \mathbb{R}$).

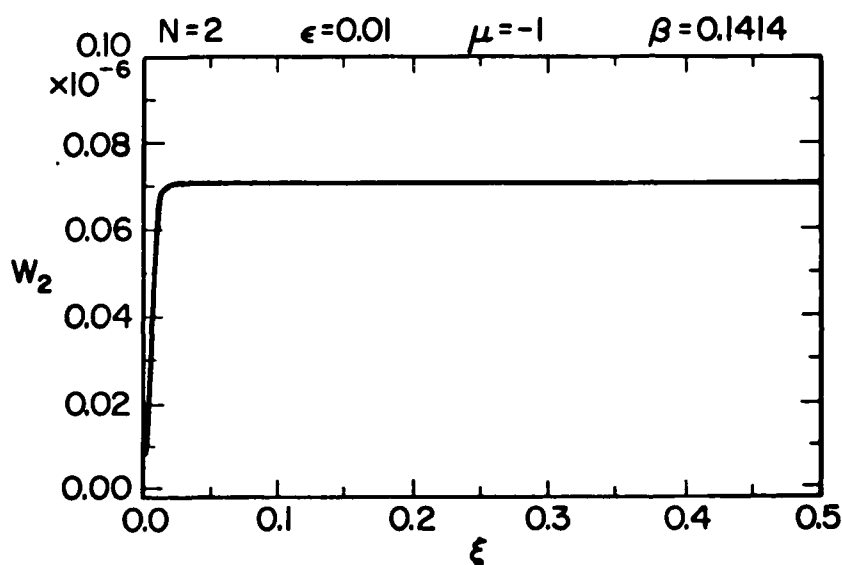


Fig. 5. Coefficient function w_2 for $\epsilon=.01$ and $\mu=-1$.

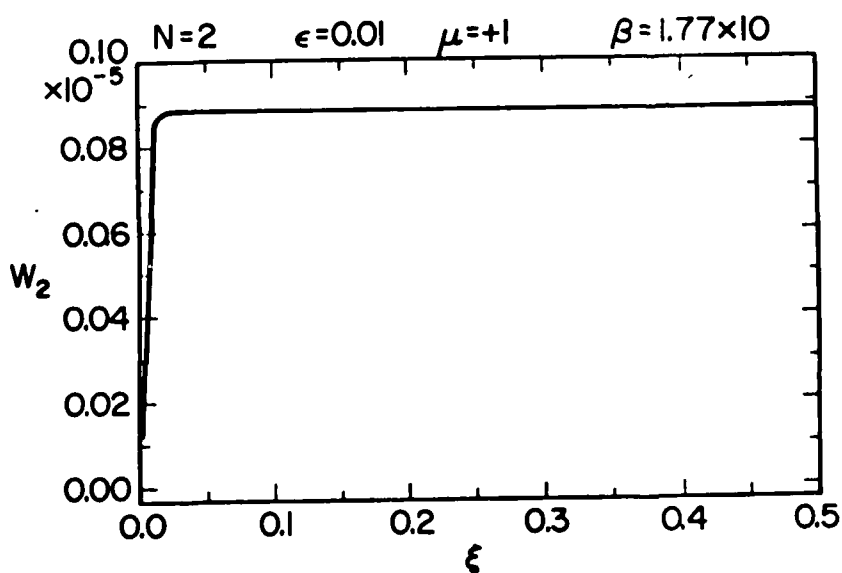


Fig. 6. Coefficient function w_2 for $\epsilon=.01$ and $\mu=+1$.

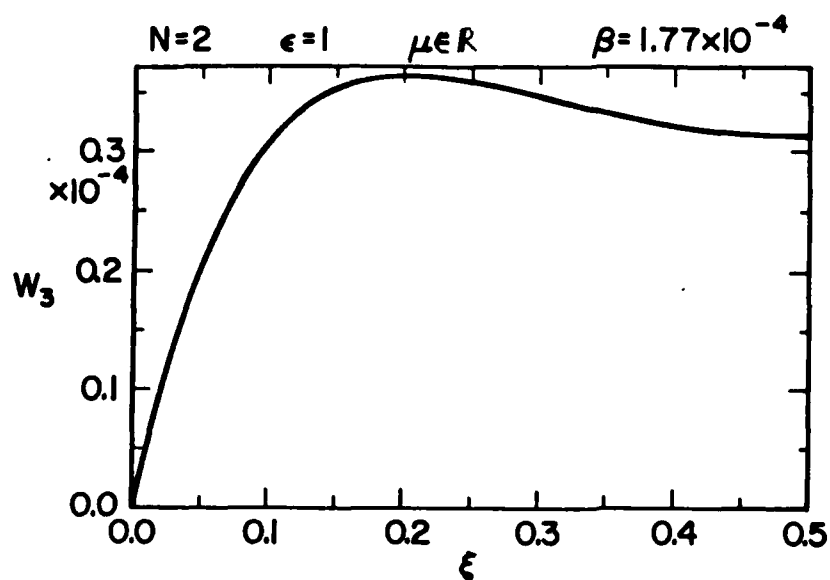


Fig. 7. Coefficient function w_3 for $\epsilon = 1$ ($\mu \in \mathbb{R}$).

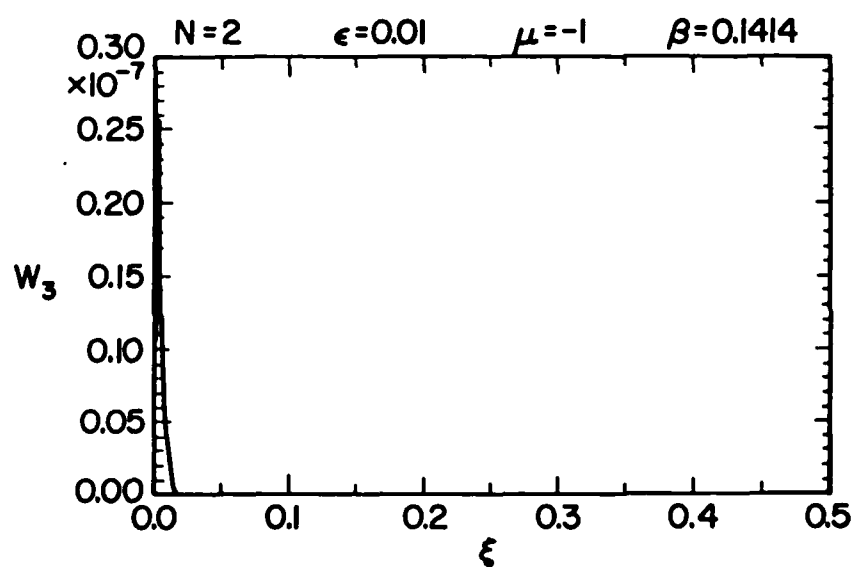


Fig. 8. Coefficient function w_3 for $\epsilon = .01$ and $\mu = -1$.

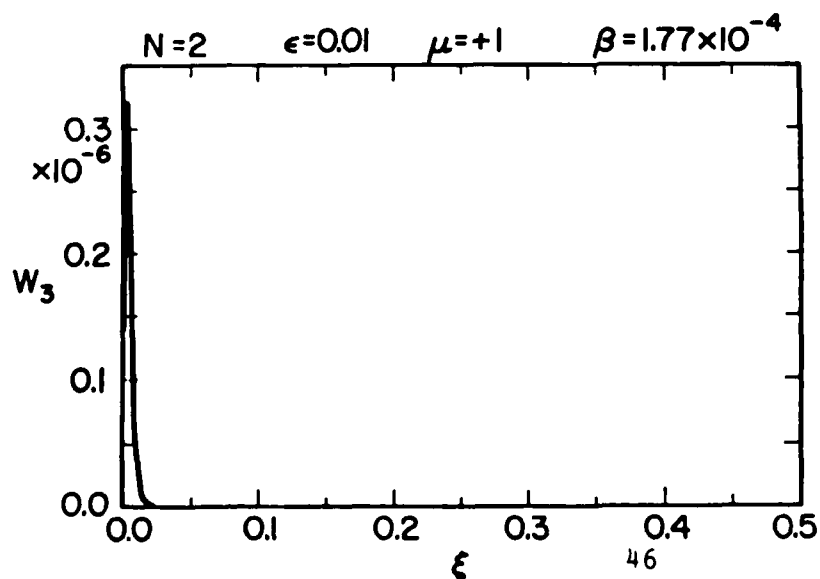


Fig. 9. Coefficient function w_3 for $\epsilon = .01$ and $\mu = +1$.

We now describe the performance of a second approach applicable for smooth, compatible β and $\mu \leq 0$. In this case, it is possible to generate an asymptotic series for the exact solution along the lines of Section 4. Taking a sufficiently large number of terms in this series, it is thus feasible to make direct error computations in the relevant norms. As an example, take $\beta = \lambda \pi^2 (1 + 3\lambda^2 \pi^2 \cos^2 \pi x) \sin \pi x$, where λ is a continuation parameter.

(This choice corresponds to taking $C_0 = \lambda \sin \pi x$, cf. Lemma 1 part (ii).)

Let the exact solution be given by (cf. Thm. 2 with subsequent remark)

$$u = \sum_{j=0}^{\infty} u^{(j)} \epsilon^{2j}$$

where each

$$u^{(j)} = \sum_{k=0}^j a_{j,2k}(\xi) \eta^{2k}, \quad j \in \mathbb{N}.$$

Denote by u_M^a the Mth partial sum in the above asymptotic series. Then

$$| |u - u_N| - |u_M^a - u_N| | \leq |u - u_M^a|,$$

where $|u - u_N|$ is sought for, $|u_M^a - u_N|$ is computable, and, by inspection

$$|u - u_M^a|_{1,2} = O(\epsilon^{2M + 3/2})$$

and

$$|u - u_M^a|_{1,4} = O(\epsilon^{2M + 5/4}).$$

From Cor. 1 and Thm. 3 it follows that

$$|u - u_N|_{1,2} = O(\epsilon^{2N + 3/2})$$

and

$$|u - u_N|_{1,4} = O\left(\epsilon^{\frac{2N + 7/4}{9} + \frac{1}{6}}\right).$$

To investigate the asymptotic rate of convergence for $\epsilon \rightarrow 0$ for $N=0$ and 1

respectively it will thus suffice to take $M=1$ and 2 (for the $W^{1,2}$ -norm) and $M=0$ (for the $W^{1,4}$ -norm).

The terms $u^{(j)}$, $j=0, \dots, M$, are generated via SMP (the symbolic manipulation program by Inference Corporation) using the recursive formulae in the proof of Thm. 2 with the twist that $\mu=0$. $|u_M^a - u_N|$ is then computed via Simpson's rule over 51 points in $[0, 1/2]$.

On the next two figures (Figs. 10 and 11) are shown the errors in the two names $W^{1,2}$ and $W^{1,4}$ for $N=0$. We see the optimal rate for $|u - u_N|_{1,2}$ and observe a rate for $|u - u_N|_{1,4}$, which, not surprisingly, is superior to the rate of Thm. 3; in fact we get the rate achievable with 1 term ($M=0$) in the asymptotic expansion.

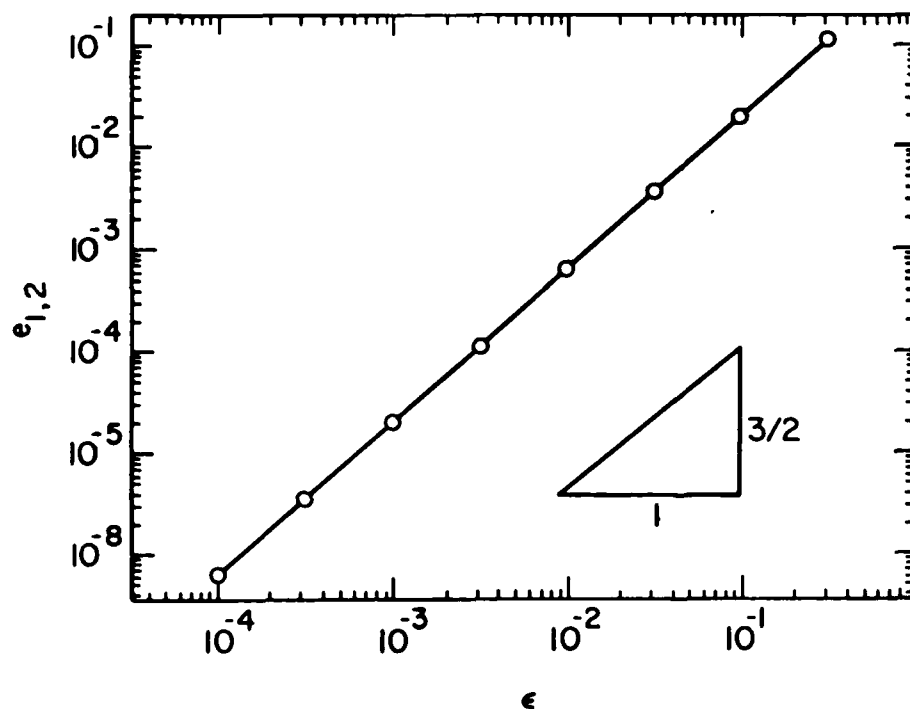


Fig 10. Error in $W^{1,2}$
- norm for $N = 0$ ($M = 1$
and 2), $\lambda = .1414$, and
 $\mu = 0$.

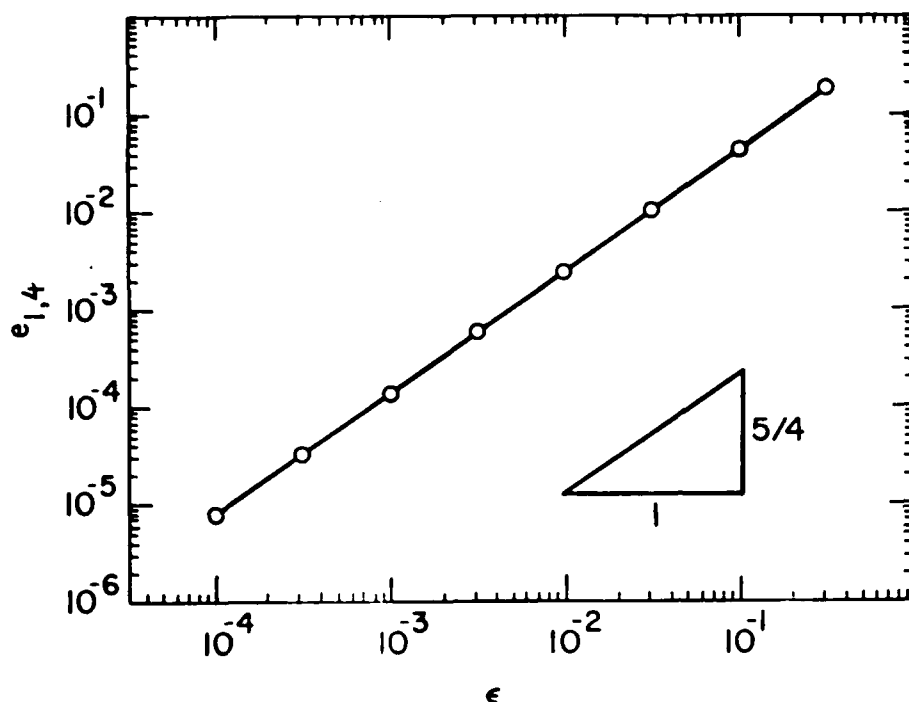


Fig. 11. Error in $W^{1,4}$
 - norm for $N = 0$ ($M = 1$
 and 2), $\psi = .1414$, and
 $\mu = 0$.

Our choice of ϕ_0 , ϕ_1 , and ϕ_2 was not ideal; it made some algebra handier, but for larger N one would need to find ψ_j out of Thm. 1 and 2 to maintain an easily invertible $A(\underline{w})$ as well as ensure that $w_j \rightarrow 0$ fast as $\epsilon \rightarrow 0$.

Computations were also carried out for other types of functions F (see(1)) exhibiting much the same behavior (although a less extensive analysis was done, see [14]) and also for the linear case (F is the identity) where it was possible to compare the CPU time spent with that of another solver: FEARS (Finite Element Adaptive Research Solver), cf. [23], solving (4) directly, and our method compared favorably (about 30 times faster; it should be noted, though, that FEARS is a more general purpose solver).

We conclude that our approach yields an efficient numerical procedure for solving (7) for ϵ small, and smooth compatible boundary data as well as for ϵ not small or in the presence of some layer in the solution.

More precisely, we would like to emphasize the following conclusions:

1. There is for domains like ours and possibly archetype domains a large savings in computational effort compared to standard nondimensional reduction solvers.
2. The approach is a general one, which no doubt can be adapted to other "stress-strain laws" as expressed through the function F in (1) as well as other types of boundary conditions.
3. The selection of basis functions $\{\psi_j\}$ is robust in at least three ways.
 - a. Even though the selection was motivated by small values of the half-thickness of the domain (ϵ), it works quite well for ϵ not small ($\epsilon = 1$ e.g.) without having to use an excessive number (N) of basis functions.
 - b. Even in the presence of nonsmooth or noncompatible loads giving rise to layers in the solution, the method works well by picking these layers up in the coefficient functions $\{v_j\}$ to the basis functions. These $v_j \rightarrow 0$ fast as $j/\epsilon \rightarrow \infty$ and thus rough input data are taken care of at fixed ϵ through increasing N at least near the layer.
 - c. Through the parameter μ we have introduced three asymptotic ranges of loads: zero ($\mu < 0$), finite ($\mu = 0$), and infinite ($\mu > 0$) limit load. We showed that one should select the same basis functions in each of the three cases. Also computationally the method works equally well in all three cases.

4. The method lends itself to adaptivity or the "feed-back" approach in at least two ways.
 - a. At fixed N , the ODE solver used is actually already adaptive, see Babuska and Majer [21] and Majer [22].
 - b. We would like to propose the following generalization of the method in this paper: let the choice of N be made locally in each subinterval of some given mesh on the basis of local error estimators. We conjecture that the computational effort thus involved would be far less than that involved in "feed-back" or adaptive methods for standard solvers since the dimension of the problem has been reduced by one.

References

1. R. Abeyaratne, "Some Finite Elasticity Problems Involving Crack-Tips" in "Modelling Problems in Crack Tip Mechanics" (ed. by J. T. Pindera), Nijhoff, Boston, 1983, pp. 3-24.
2. S.S. Antman and G. Rosenfeld, "Global Behaviour of Buckled States of Nonlinearly Elastic Rods," SIAM J. Review, v. 20, 1978, pp. 513-566; "Corrections and Additions", SIAM J. Review, v. 22, 1980, pp. 186-187.
3. I. Babuška, B. Szabo, and I.N. Katz, "The p-version of the Finite Element Method," Report WU/CCM-79/1, Washington University in St. Louis, May 1981, SIAM J. Num. Anal.
4. J. Bergh and J. Löfström, "Interpolation Spaces, An Introduction", Springer-Verlag, Berlin, 1976.
5. C. Canuto and Quarteroni, "Approximation Results for Orthogonal Polynomials in Sobolev Spaces," Math. Comp., v. 38, pp. 67-86.
6. P.G. Ciarlet and P. Destuynder, "A justification of the two-dimensional linear plate model," J. Mécanique, v. 18, 1979, pp. 315-344.
7. P.G. Ciarlet, "The Finite Element Method for Elliptic Problems", North-Holland, Amsterdam, 1980.
8. K.O. Friedrichs, "The edge effect in the bending of plates," Reissner Anniv. Vol., pp. 197-210, Edwards, Ann Arbor, Mich., 1949, "Kirchoff's boundary conditions and the edge effect for elastic plates," Proc. Symp. in Appl. Math., v. 3, 1950, pp. 117-124, McGraw-Hill, New York.

9. K.O. Friedrichs and R.F. Dressler, "A boundary layer theory for elastic bending of plates," *Comm. Pure Appl. Math.*, v. 14, 1961, pp. 1-33.
10. R. Glowinski and A. Marocco, "Sur l'approximation par éléments finis d'ordre un, et la résolution par pénalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires". *Rev. Franc. Automat. Informat. Recherche Opér. Sér Rouge Anal. Numer.*, v. R-2, 1975, pp. 41-76.
11. A.L. Gol'denweizer, "Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity," *J. Appl. Math. Mech.*, v. 26, 1962, pp. 1000-1025.
12. M.E. Gurtin and R. Temam, "On the anti-plane shear problem in finite elasticity," *J. Anal. Elasticity*, v. 11, 1981, pp. 197-206.
13. D. Jackson, "The Theory of Approximation", AMS Colloquium Publications, v. 11, 1930.
14. S. Jensen, "Dimensional Reduction for Nonlinear Boundary Value Problems", Ph.D. Thesis, July 1985, University of Maryland.
15. L.M. Kachanov, "The Basic Theory of Plasticity", Gostekhizdat, Moscow, 1956.
16. J.K. Knowles and E. Sternberg, "Large Deformations Near the Tips of an Interface-Crack Between Two Neo-Hookean Sheets", *J. of Elasticity*, v. 13, 1983, p. 257.
17. J.K. Knowles and E. Sternberg, "On the Singularity Induced by Certain Mixed Boundary Conditions in Linearized and Nonlinear Elastostatics", *Int. J. Solids and Structures*, v. 11, 1975, p. 1173.

18. P.P. Korovkin "Inequalities", Blaisdel, New York, 1961.
19. R. Kosler, " $W^{m,2}$ -Error Estimates of Galerkin Approximations of Weak Elliptic Quasilinear Boundary Value Problems of Order $2m$ " (to appear), 1985.
20. A. Langenbach, "Variationsmethoden in nichtlinearen Elastizität- und Plastizitätstheorie", Wiss. Z. Humbolt-Univ., Berlin, Math.-Nat. R. IX.
21. I. Babuška and V. Majer, "The Factorization Method for the Numerical Solution of Linear Two Point Boundary Value Problems for ODE's", SIAM J. on Num. An. (to appear), 1986.
22. V. Majer, "Numerical Solution of Boundary Value Problems for Ordinary Differential Equations of Nonlinear Elasticity," Ph.D. Thesis, Univ. of Maryland, 1984.
23. G. Meinardus, "Approximation von Funktionen und ihre numerische Behandlung", Springer-Verlag, Berlin, 1964.
24. C. Mesztenyi and W. Szymczak, "FEARS user's manual for UNIVAC 1100," Tech. Note BN-991, IPST, Univ. of Maryland, College Park, 1982.
25. C.B. Morrey, Jr., "Multiple integrals in the calculus of variations", Springer-Verlag, Heidelberg, 1966.
26. P.M. Naghdi, "The Theory of Shells and Plates" in "Mechanics of Solids", v. II, Springer-Verlag, Berlin, 1984.
27. A. Pinkus, "N-widths in Approximation Theory", Springer, 1985.
28. E.L. Reiss and S. Locke, "On the theory of plane stress," Quart. Appl. Math., v. 19, 1961, pp. 195-203.

29. M.I. Višik, "Quasi-linear strongly elliptic systems of differential equations in divergence form," Proc. of the Steklov Inst., 1962, Moscow, pp. 140-208.
30. M. Vogelius and I. Babuška, "On a Dimensional Reduction Method I. The Optimal Selection of Basis Functions," Math. Comp., v. 37, 1981, pp. 31-46.
31. M. Vogelius and I. Babuška, "On a Dimensional Reduction Method II. Some Approximation-theoretic Results," Math. Comp., v. 37, 1981, pp. 47-68.
32. M. Vogelius and I. Babuška, "On a Dimensional Reduction Method III. A Posteriori Error Estimation and an Adaptive Approach," Math. Comp., v. 37, 1981, pp. 361-384.

The Laboratory for Numerical analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- o To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.
- o To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.
- o To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.
- o To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Bureau of Standards.
- o To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.)

Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

END

12-86

DTIC